# Introduction to Delta Method in Econometrics

Alexey Khazanov

Lomonosov Moscow State University, Economics department; New Economic School

# Instead of Introduction

1. **What is Delta method used for?**
   Asy distribution of $\hat{\theta} \rightarrow$ asy distribution of $g(\hat{\theta})$
2. **What kind of theory is behind the Method?**
   CLT, Slutsky (CMT), Taylor expansion
3. **What is the alternative?**
   Bootstrap

References: B. Hansen's "Econometrics".

# Table of Contents

# Theoretical Background - 1

Theorem (**Central Limit Theorem (CLT)** )

*Let* $\{z_n\}$ *be IID with* $\mathbb{E}[z_i] = \mu$ *and* $\mathbb{V}[z_i] = \sigma^2$. *Then,*

$$\sqrt{n}\left(\frac{1}{n}\sum z_i - \mu\right) \xrightarrow{d} N(0, \sigma^2),$$

*as* $n \to \infty$. *Lindberg-Levy version of the theorem for IID obs.*

# Theoretical Background - 1

Theorem (**Central Limit Theorem (CLT) )**

*Let* $\{z_n\}$ *be IID with* $\mathbb{E}[z_i] = \mu$ *and* $\mathbb{V}[z_i] = \sigma^2$*. Then,*

$$\sqrt{n}\left(\frac{1}{n}\sum z_i - \mu\right) \xrightarrow{d} N(0, \sigma^2),$$

*as* $n \to \infty$*. Lindberg-Levy version of the theorem for IID obs.*

Theorem (**Continuous Mapping Theorem (CMT))**

*If* $z_n \xrightarrow{d} z$ *as* $n \to \infty$ *and* $g : \mathbb{R}^m \to \mathbb{R}^k$ *has the set of discontinuity points* $D_g$ *such that* $Pr(z \in D_G) = 0$*, then* $g(z_n) \xrightarrow{d} g(z)$ *as* $n \to \infty$

# Theoretical Background - 2

**Theorem (Slutsky's Theorem )**

*If* $z_n \xrightarrow{d} z$ *and* $c_n \xrightarrow{p} c$ *as* $n \to \infty$, *then*

1. $z_n + c_n \xrightarrow{d} z + c$
2. $z_n c_n \xrightarrow{d} zc$
3. $\frac{z_n}{c_n} \xrightarrow{d} \frac{z}{c}$ *if* $c \neq 0$

# Theoretical Background - 2

**Theorem (Slutsky's Theorem)**

*If* $z_n \xrightarrow{d} z$ *and* $c_n \xrightarrow{p} c$ *as* $n \to \infty$, *then*

1. $z_n + c_n \xrightarrow{d} z + c$
2. $z_n c_n \xrightarrow{d} zc$
3. $\frac{z_n}{c_n} \xrightarrow{d} \frac{z}{c}$ *if* $c \neq 0$

**Taylor's expansion**

Assume g(x) is continuous and twice differentiable for any $x \in X$. Then, for some $x_0 \in X$

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + \frac{1}{2!}g''(x_0)(x - x_0)^2 + o(x^2)$$

# Table of Contents

# Deriving Univariate Delta method

Let $\sqrt{n}(\hat{\mu} - \mu) \xrightarrow{d} \xi$. What is the asymptotic distribution of $g(\hat{\mu})$ ?

1. Apply Taylor Expansion at $\mu$

$$g(\hat{\mu}) = g(\mu) + g'(\mu)(\mu - \mu) + o(\hat{\mu})$$

2. Re-arrange the terms

$$g(\hat{\mu}) - g(\mu) = g'(\mu)(\mu - \mu) + o(\hat{\mu})$$
$$\sqrt{n}\,(g(\hat{\mu}) - g(\mu)) = g'(\mu)\sqrt{n}(\mu - \mu) + \sqrt{n}o(\hat{\mu})$$

3. Use $\sqrt{n}(\hat{\mu} - \mu) \xrightarrow{d} \xi$ Then,

$$\sqrt{n}\,(g(\hat{\mu}) - g(\mu)) \xrightarrow{d} g'(\mu)\xi$$

Assume $\xi \sim N(0, \sigma^2)$. Then,

$$\sqrt{n}\,(g(\hat{\mu}) - g(\mu)) \xrightarrow{d} N(0, (g'(\mu))^2\,\sigma^2)$$

# Multivariate Delta Method

**Theorem (Delta Method)**

*If* $\sqrt{n}(\hat{\mu} - \mu) \xrightarrow{d} \xi$, *where* $g(u)$ *is continuously differentiable in a neighborhood of* $\mu$ *then as* $n \to \infty$

$$\sqrt{n}\,(g(\hat{\mu}) - g(\mu)) \xrightarrow{d} G'\xi,$$

*where* $G(u) = \frac{\partial}{\partial u} g(u)'$ *and* $G = G(\mu)$. *In particular, if* $\xi \sim N(0, V)$, *then as* $n \to \infty$

$$\sqrt{n}\,(g(\hat{\mu}) - g(\mu)) \xrightarrow{d} N(0, G'VG)$$

# Multivariate Delta Method

**Theorem (Delta Method (short))**

*If* $\sqrt{n}(\hat{\mu} - \mu) \xrightarrow{d} N(0, V)$, *then* $\sqrt{n}\left(g(\hat{\mu}) - g(\mu)\right) \xrightarrow{d} N(0, G'VG)$ *where* $G(u) = \frac{\partial}{\partial u} g(u)'$ *and* $G = G(\mu)$.

## Examples

For all examples from the regression analysis assume OLS post-estimation results under IID assumption about the error term.

1. $\hat{\mu}$ is some unbiased estimate for $\mu$ – population mean. Find asymptotic distribution for $log(\hat{\mu})$ and $exp(\hat{\mu})$
2. Find confidence interval for the top of the parabola estimate.
3. $\hat{\mu}$ is some unbiased estimate for $\mu$ – population mean. Find asymptotic distribution for $\hat{\mu}^2$ if $\mu = 0$

# Table of Contents

# Second-Order Delta Method

**Theorem (Second-Order Delta Method)**

*If $\sqrt{n}(\hat{\mu} - \mu) \xrightarrow{d} N(0, \sigma^2)$, where $g(u)$ is continuously differentiable in a neighborhood of $\mu$, and $g'(\mu) = 0$ and if $g''(\mu) \neq 0$ then as $n \to \infty$*

$$n(g(\hat{\mu}) - g(\mu)) \xrightarrow{d} \sigma^2 \frac{g''(\mu)}{2} \chi_1^2$$

Now, consider the previous example.
$\hat{\mu}$ is some unbiased estimate for $\mu$ – population mean. Find asymptotic distribution for $\hat{\mu}^2$ if $\mu = 0$.

# Table of Contents

# Usage

- In Statistics
- In Econometrics
- Bootstrap vs. Delta Method

# The _Thank_you_ slide