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THE AGE PATTERN OF MORTALITY

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1. INTRODUCTION

1.1. The development of a 'law of mortality', a mathematical expression for the graduation of the age pattern of mortality, has been of interest since the development of the first life tables by John Graunt (1662) and Edmund Halley (1693). Although Abraham De Moivre proposed a very simple law as early as 1725 the best known early contribution is probably that of Benjamin Gompertz (1825). Since World War II mathematical formulae have been used to graduate sections of the English Life Tables, as well as assured lives mortality[7], and pensioner and annuitant mortality[8]. Reviews of attempts at finding the 'law of mortality' have been given by Elston[6] and Benjamin and Haycocks[4].

1.2 A mortality graduation can only be considered successful if the graduated rates progress smoothly from age to age and at the same time they reflect accurately the underlying mortality pattern. In other words, the graduation must smooth out irregularities due to random variation and age misstatement while maintaining all the essential underlying variations in the mortality pattern. In this paper we propose a mathematical expression or 'law of mortality' which we fit to post-war Australian national mortality data.

2. THE BASIC CURVE AND ITS PARAMETERS

2.1 The basic curve we suggest is

$$q_x/p_x = A^{(x+B)^C} + De^{-E(\ln x - \ln F)^2} + GH^x \tag{1}$$

where q_x is the probability of dying within 1 year for a person aged x exactly, and $p_x = (1 - q_x)$. $A, B, \ldots, H$ are parameters to be estimated. The curve, which is not unlike that of Thiele[10], is continuous and applicable over the entire age range $0 \le x < \infty$, and allows q_x to take values between zero and one only. It has relatively few parameters, all of which have demographic interpretations. Furthermore, the curve is sufficiently flexible to fit a wide variety of mortality patterns.

2.2 The mathematical formula contains three terms, each representing a distinct component of mortality. The first, a rapidly declining exponential, reflects the fall in mortality during the early childhood years as the child adapts to its new environment and gains immunity from the diseases of the outside world. This component of mortality has three parameters. A, which is nearly equal to q_1, measures the level of mortality. C measures the rate of mortality decline in childhood (the rate at which a child adapts to its environment). The higher the value of C, the faster mortality declines with increasing age. B is an age displacement to account for infant mortality. When $B=0$, q_0 equals ·5 no matter what values A and C may have; and for fixed C, the higher the value of B, the closer q_0 is to q_l. B, therefore, measures the location of infant mortality within the range $(q_l, \frac{1}{2})$. In practice, B is close to zero (taking values between ·01 and ·03 in the postwar Australian experience, for example), and its effects on the graduated mortality rates at ages other than zero are negligible.

2.3 The third term in the formula, the well-known Gompertz exponential, reflects the near geometric rise in mortality at the adult ages, and is generally considered to represent the ageing or deterioration of the body, i.e. senescent mortality. The parameter G represents the base level of senescent mortality while H reflects the rate of increase of that mortality. Technically, G represents senescent mortality at age zero. The third term can perhaps be expressed more meaningfully as H^{x-x_0} where x_0 is the age at which $q_x/p_x=1$ or $q_x=\cdot5$. Admittedly, this age is very close to the end of the life table where the observed rates themselves are less certain, but as a senescence parameter it is more readily interpretable than G.

2.4 The remaining term, a function similar to the lognormal, reflects accident mortality for males and accident plus maternal mortality for the female population; i.e. additional mortality superimposed on the 'natural curve of mortality' as described by the other two terms. The 'accident hump' is found in all populations, and appears either as a distinct hump in the mortality curve or at least as a flattening out of the mortality rates, generally between ages 10 and 40. The accident term in (1) has three parameters: F indicating location, E representing spread, and D the severity.

2.5 The use of the ratio q_x/p_x on the left-hand side of (1) has the advantage that q_x must lie between 0 and 1 for all x, and the fit at the older ages is also improved[7]. The three components of mortality and their contribution to total mortality are illustrated graphically in Figure 1, using Australian national mortality 1970–72 (males).

3. APPLICATION TO AUSTRALIAN NATIONAL MORTALITY DATA

3.1 We now demonstrate the use of the formula by graduating post-war Australian national mortality. The data are deaths by age and sex for the three-year periods 1946–48, 1960–62, and 1970–72 and the mid-year census populations by age and sex for 1947, 1961, and 1971[1,2,3] Central mortality rates by age for each sex were calculated as

$$m_x=\theta_x/3P_x$$

where θ_x is the number of deaths during the three-year period for persons aged x and P_x is the appropriate mid-year census population aged x. The central mortality rates were transformed to q_x values by the classical formula

$$q_x=2m_x/(2+m_x)$$

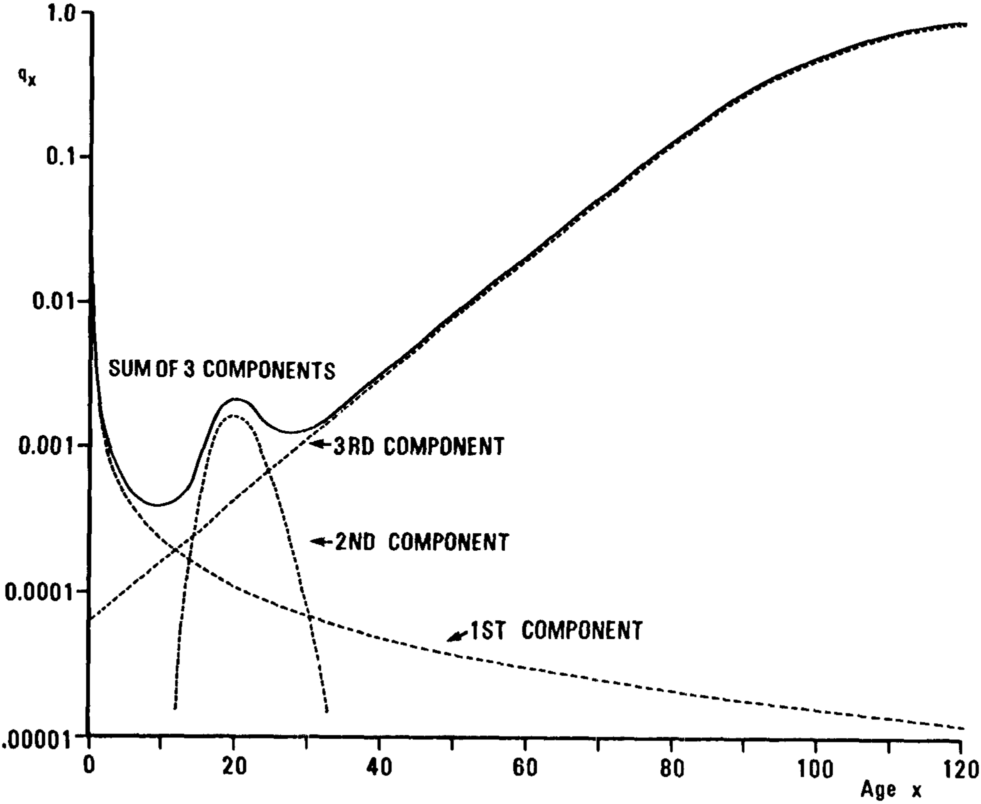

Figure 1. The graduated q_x curve and its three components: Australian national mortality, 1970–72 (males).

3.2 The parameters of the curve were estimated by least squares using Gauss-Newton iteration[9]. The function minimized was

$$S^2 = \sum_{x=0}^{85} \left(\frac{q_x}{\mathring{q}_x} - 1{\cdot}0\right)^2$$

where q_x is fitted value at age x and $\mathring{q}_x$ is the observed mortality rate. That is, the sum of the squares of the proportional difference between the fitted and observed rates was minimized. The observed rates above age 85 were excluded from the calculation because they appeared to be less reliable.

3.3 The observed and graduated mortality rates for Australian males and females 1946–48, 1960–62, and 1970–1972 are presented graphically in Figures 2 to 7 and listed in the Appendix. As indicated above the fitted values for ages past

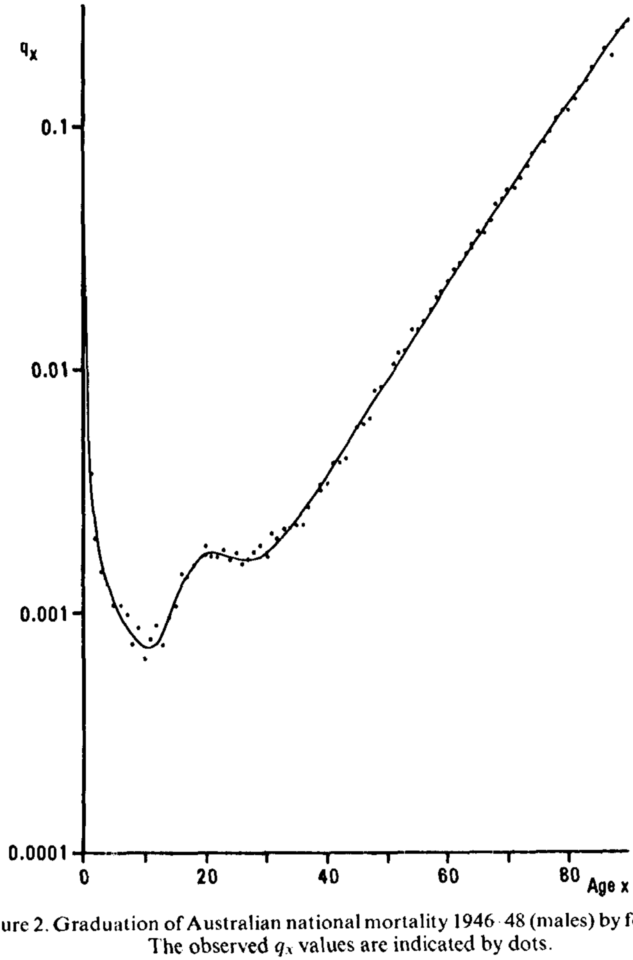

Figure 2. Graduation of Australian national mortality 1946–48 (males) by formula (1). The observed q_x values are indicated by dots.

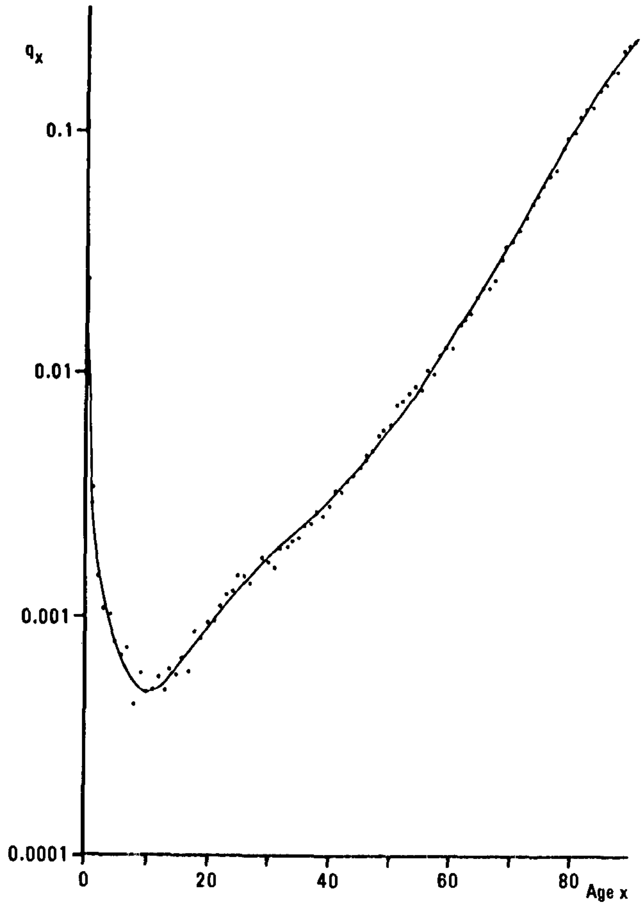

Figure 3. Graduation of Australian national mortality 1946–48 (females) by formula (1). The observed q_x values are indicated by dots.

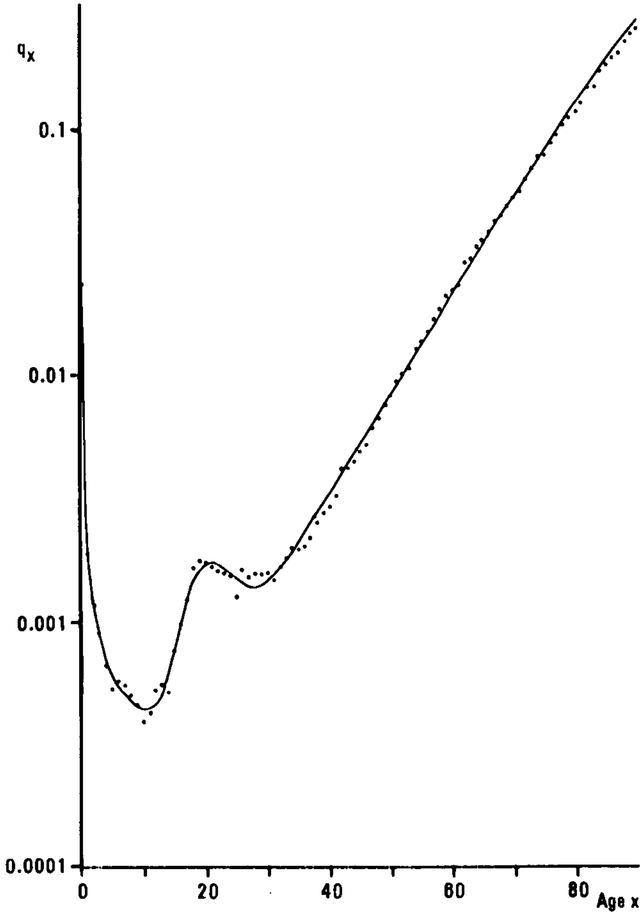

Figure 4. Graduation of Australian national mortality 1960–62 (males) by formula (1). The observed q_x values are indicated by dots.

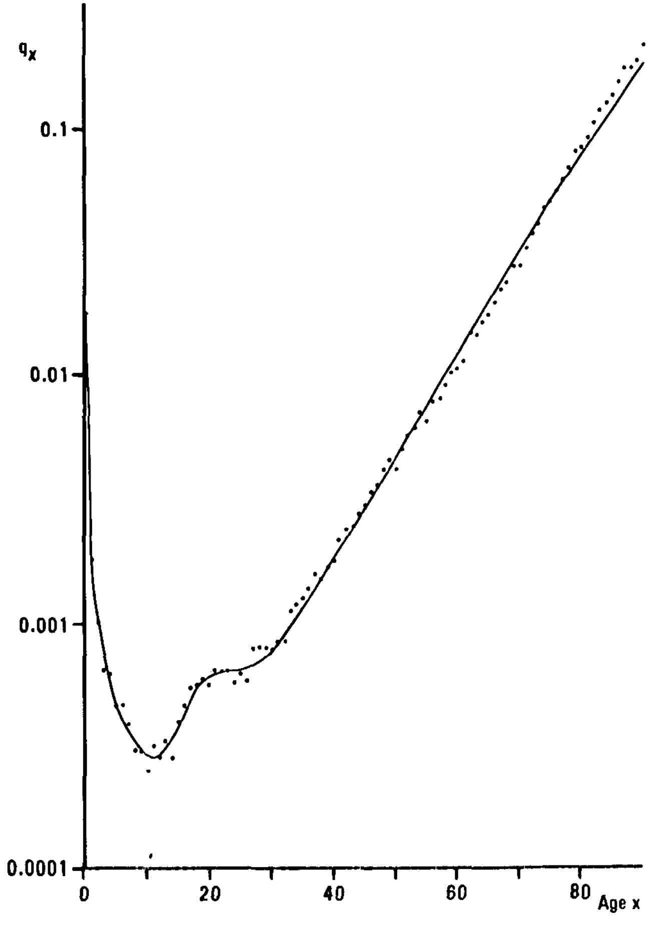

Figure 5. Graduation of Australian national mortality 1960–62 (females) by formula (1). The observed q_x values are indicated by dots.

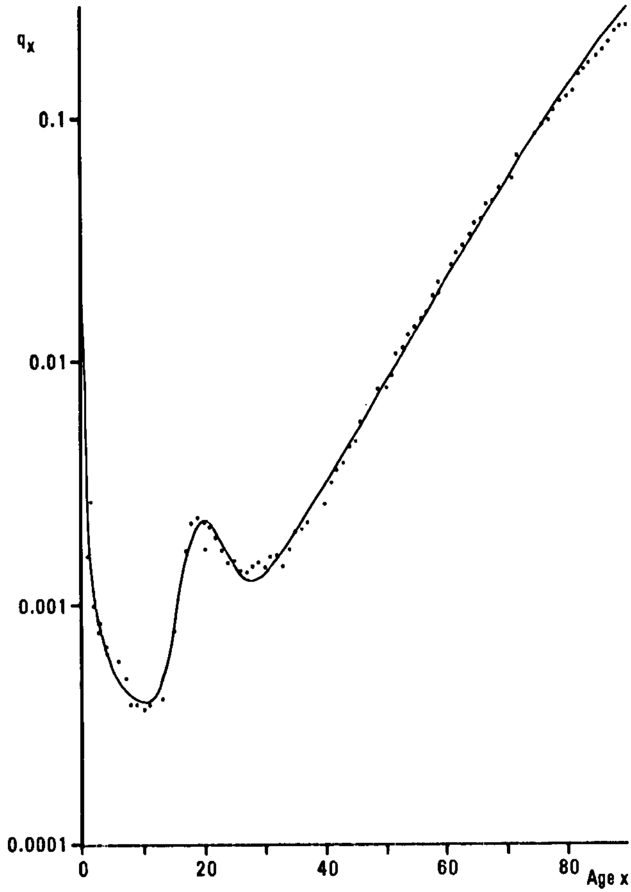

Figure 6. Graduation of Australian national mortality 1970–72 (males) by formula (1). The observed q_x values are indicated by dots.

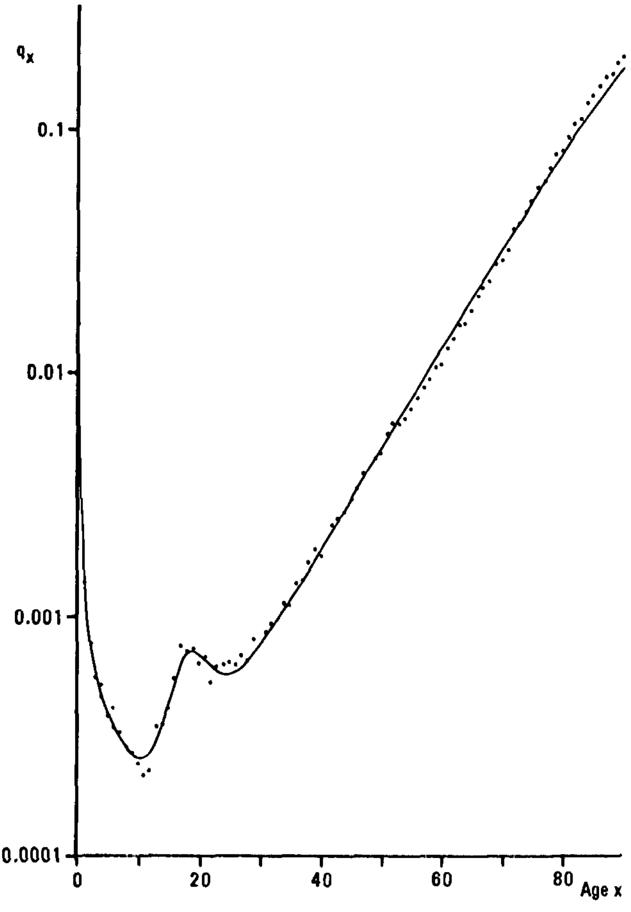

Figure 7. Graduation of Australian national mortality 1970–72 (females) by formula (1). The observed q_x values are indicated by dots.

85 are extrapolations. The graduated mortality probabilities appear to provide adequate representations of the age patterns of mortality in all six cases whether or not the 'accident hump' is strong and prominant (as for the 1970–72 experience) or nearly non-existent (as for the 1946–48 female experience), although a few minor defects are evident. Where the 'accident hump' is prominent the graduation appears to overstate mortality on part of the downward portion of the curve and understate it for a few years thereafter. Furthermore, a small amount of curvature, not accounted for by the Gompertz term, is evident in the data for the older adult ages. Two variations of the basic graduation formula to deal with this curvature are presented later in Section 4. Given the relatively few parameters used, however, the complex shape of the mortality curve itself, the large number of ages and the variety of mortality patterns covered, the formula appears promising.

3.4 The parameters for the graduations in Figures 2 to 7 are presented in Table 1. Parameters A, B, and C describe the pattern of mortality during early childhood. A has fallen by over 50% from 1946–48 to 1970–72 indicating that for both males and females child mortality has declined considerably. The drop in mortality after age 1, indicated by C, has become less pronounced in recent years. In other words, mortality has been declining faster in recent years at the younger than at the older childhood ages. For both males and females the parameter B has also been falling. Relative to q_l, therefore, infant mortality has not been declining as quickly as mortality at the other childhood ages. Throughout the period males have experienced higher child mortality than females (parameter A). Female child mortality has not only been lower but the rate of mortality decline with age has been faster throughout the 25-year period (females have higher values of C). The higher B values for females indicate that females had relatively lower infant mortality within their $(q_l, 1)$ interval.

3.5 The second term in (1) represents the 'accident hump'. For males during the entire period from 1946–48 to 1970–72 and for females from 1960–62 only the contribution of the accident hump to mortality has increased considerably

Table 1. *Parameters for model (1). Australian national mortality 1946–48, 1960–62, and 1970–72.*

	Males			Females		
Parameter	1946–48	1960–62	1970–72	1946–48	1960–62	1970–72
A	·00341	·00184	·00163	·00293	·00177	·00137
B	·0208	·0189	·0144	·0336	·0304	·0251
C	·1284	·1189	·1182	·1339	·1309	·1249
D	·00094	·00110	·00164	·00156	·00025	·00039
E	9·49	13·55	18·49	1·29	8·83	16·80
F	20·22	20·43	19·88	53·17	20·37	18·58
G	·0000862	·0000711	·0000643	·0000196	·0000353	·0000383
(x_0)	(101·1)	(101·0)	(100·0)	(101·7)	(105·4)	(106·0)
H	1·0970	1·0992	1·1013	1·1136	1·1022	1·1007

(parameter D). Accident mortality is now concentrated in a narrower band of ages, and as a result E has increased. The location of the hump however has remained more or less constant near age 20 (parameter F). When we compare the 'accident hump' for males and females for the period 1960–62 and 1970–72 we see that in both periods males experienced considerably higher 'accident' mortality, with D taking values about four times greater for males than for females. The male hump is also narrower (E) but the differential is small and may be declining. The location of the accident hump (F) may be at a slightly younger age for females than for males.

3.6 We have avoided discussing the female accident hump for the 1946–48 period. The 1946–48 female parameters D, E, and F clearly indicate that the second term in (1) is describing something other than the 'accident hump' as we have defined it. The fitted hump is centred at age 53, has a very large spread, and is rather severe. The fit throughout the entire age range is however very good (Figure 3). The usual accident hump is in fact nearly non-existent for Australian females 1946–48 and, as a result, the least squares fitting procedure found another 'hump' later in the life span. The female q_x/p_x curve, when plotted on logarithmic paper, retains some curvature at the older ages and does not become a straight line. The unusual 'accident hump' adjusts for this curvature. We return to this problem again in Section 4.

3.7 The third term in (1) represents the ageing of the body (senescent mortality), and its parameters describe the age pattern of mortality at the older ages. It is clear that with advancing age $\ln(q_x/p_x)$ approaches the straight line passing through $(0, \ln G)$ and $(x_o, 0)$. The base senescent parameter G for males has consistently declined throughout the period, albeit slightly from 1960–62; at the same time x_o (the age at which the mortality rate q_x equals ·5) has fallen. The net effect has been a slight tilting upwards of the $\ln(q_x/p_x)$ curve. In other words the curve has become somewhat steeper and the slope of the line, $\ln H$, confirms this. The same tilting is evident in the $\ln q_x$ curve. For females there is some indication of a rise in G since 1960–62 and a rise in x_o, so that the $\ln(q_x/p_x)$ curve has tilted downwards slightly, and $\ln H$ confirms this. For both sexes the net effect of the tilting on mortality is small. The values of G were higher and the values of x_o lower for males than for females throughout the period, indicating higher male mortality throughout the senescent age span.

4. VARIATIONS OF THE BASIC MORTALITY CURVE

4.1 According to (1) $\ln(q_x/p_x)$ becomes a straight line with advancing age. We have already noted in Section 3, however, that for Australian females 1946–48 the least-squares fitting procedure produced an unusual 'accident hump' centred at age 53 and spread over a wide range. The true 'accident hump' near age 20 was almost non-existent for this experience, and as a result the computer algorithm fitted a hump at the older ages to account for the curvature which remained in the $\ln(q_x/p_x)$ curve at those ages. A small amount of curvature is also evident in the other data we examined, and we therefore sought a variation of the basic curve (1) which would deal satisfactorily with the problem.

4.2 The mortality curve (1) is almost indistinguishable from

$$q_x = A^{(x+B)^C} + De^{-E(\ln x - \ln F)^2} + GH^x/1 + GH^x \quad (1a)$$

In this form the dependent variable itself is the variable of interest and the three components of mortality are additive. However, q_x may theoretically (although probably never in actual practice) assume values greater than unity.

4.3 Because of the residual curvature evident in the data we examined the following alternative models were studied:

$$q_x = A^{(x+B)^C} + De^{-E(\ln x - \ln F)^2} + GH^x/(1 + KGH^x) \quad (2)$$

$$q_x = A^{(x+B)^C} + De^{-E(\ln x - \ln F)^2} + GH^{x^K}/1 + GH^{x^K}. \quad (3)$$

The observed rates and those graduated by (2) and (3) are presented graphically in Figures 8 to 19. The actual rates are listed in the Appendix.

4.4 Both variations lead to improvements in the fit, especially at the older ages. For Australian males (where the curvature of ln (q_x/p_x) is concave downward) formula (3) provides a better fit than (2); however, in the case of the females (where curvature is concave upward) the reverse is true. Although (2) provides a good graduation over the fitted age range the formula cannot always be used over the entire age span. This is because the lim $q_x = 1/K$ as $x \to \infty$, and the limit may lie outside the interval (0, 1), in which case an alternative definition of q_x must be adopted beyond a certain age.

4.5 Analysis of the parameter values using (2) and (3) (Tables 2 and 3) shows that with few exceptions the interpretations described previously are still valid and the time changes and sex differences remain. With the introduction of the ninth parameter to allow for curvature at the older ages the 'accident hump' for the 1946–48 females returns to something closer to its original meaning, and the interpretations of G and H are no longer clouded by an unusual accident hump.

Table 2. *Parameters for model (2). Australian national mortality 1946–48, 1960–62, and 1970–72.*

	Males			Females		
Parameter	1946–48	1960–62	1970–72	1946–48	1960–62	1970–72
A	·00337	·00181	·00160	·00288	·00182	·00142
B	·0202	·0106	·00112	·0410	·0410	·0350
C	·1256	·1130	·1112	·1409	·1402	·1345
D	·00096	·00111	·00163	·00059	·00023	·00038
E	8·71	11·85	16·71	3·88	17·02	21·86
F	20·41	20·66	20·03	28·82	19·26	18·27 ·
G	·0000752	·0000570	·0000502	·0000735	·0000519	·0000507
H	1·1002	1·1044	1·1074	1·0910	1·0932	1·0937
K	1·781	2·265	2·416	−2·398	−3·176	−2·800

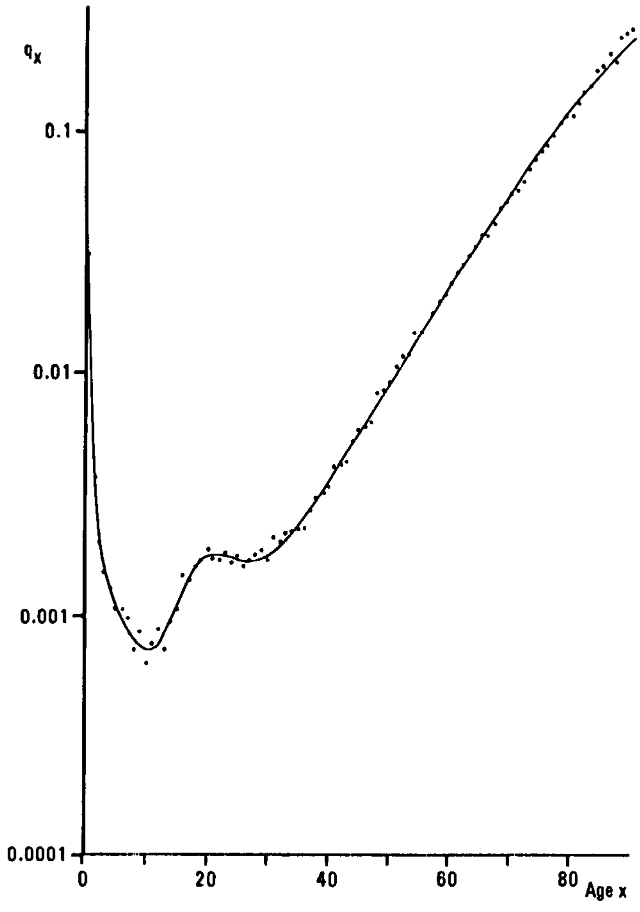

Figure 8. Graduation of Australian national mortality 1946–48 (males) by formula (2). The observed q_x values are indicated by dots.

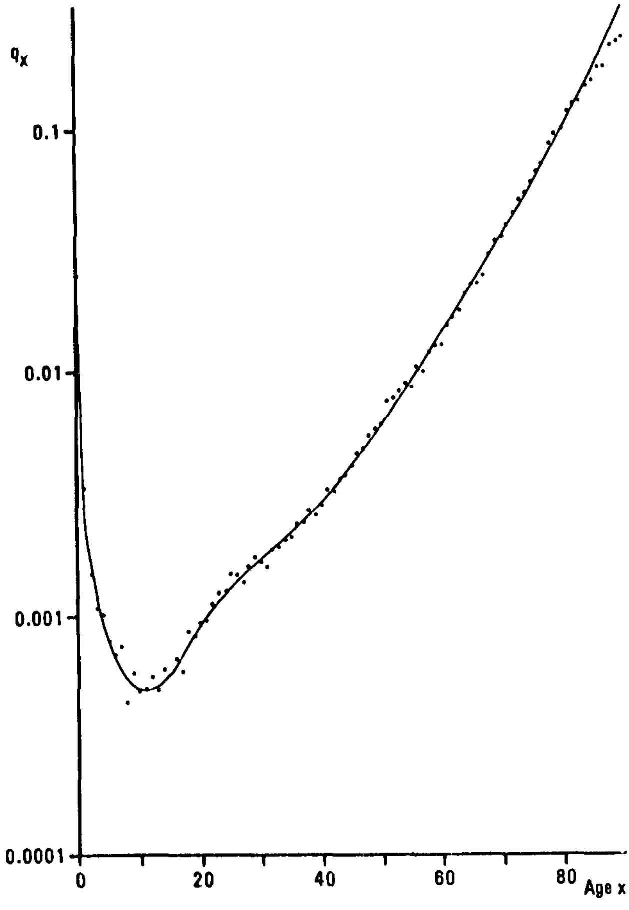

Figure 9. Graduation of Australian national mortality 1946–48 (females) by formula (2). The observed q_x values are indicated by dots.

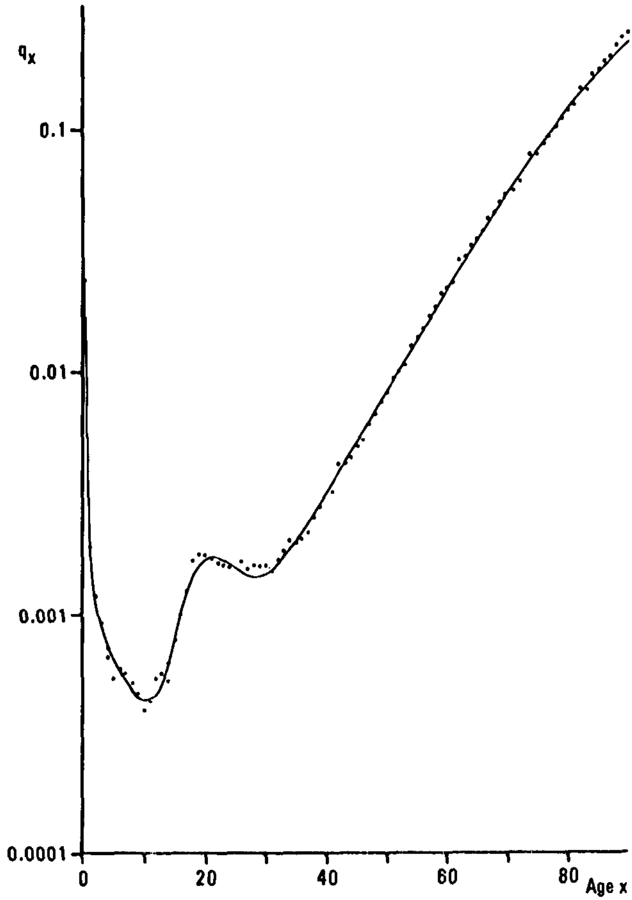

Figure 10. Graduation of Australian national mortality 1960 62 (males) by formula (2). The observed q_x values are indicated by dots.

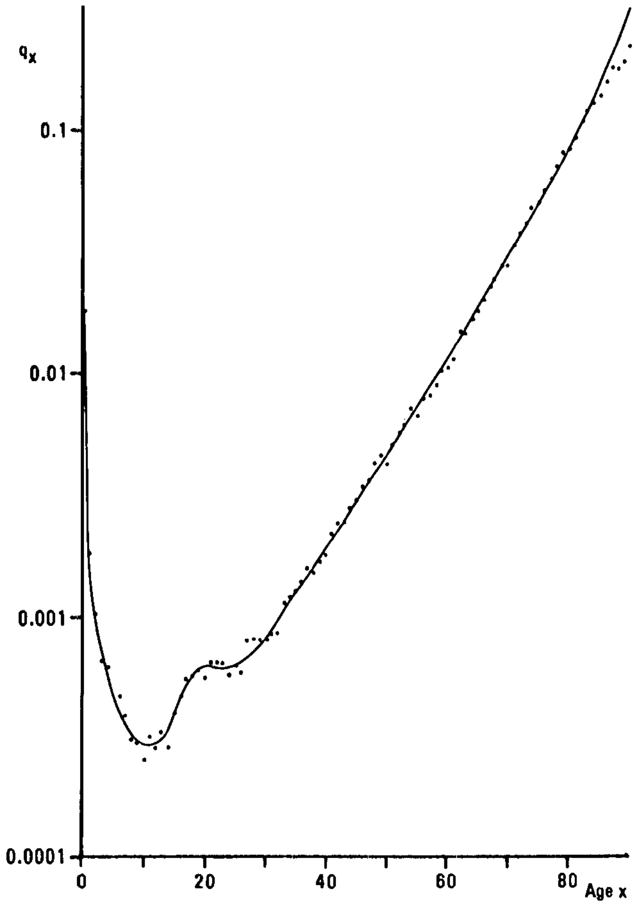

Figure 11. Graduation of Australian national mortality 1960–62 (females) by formula (2). The observed q_x values are indicated by dots.

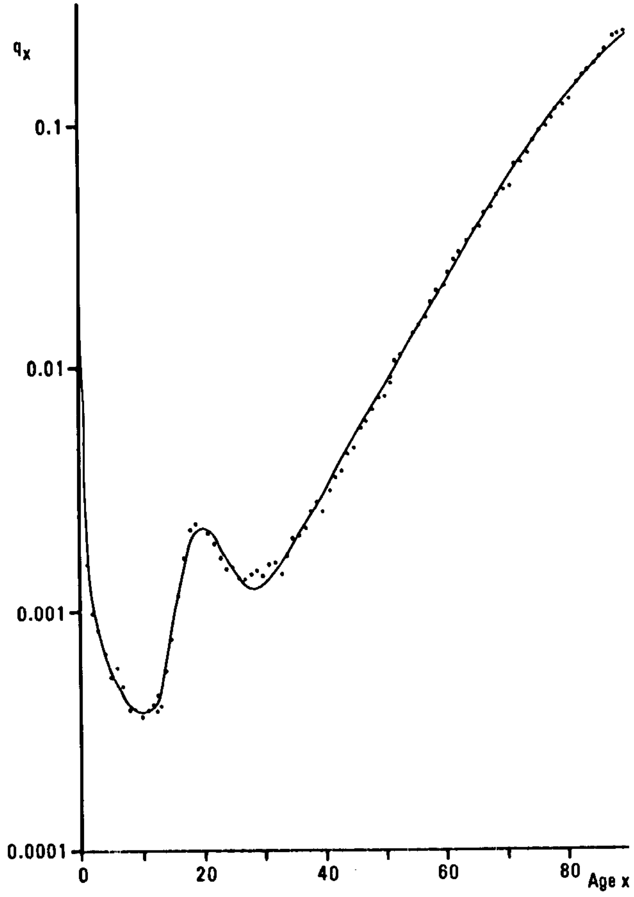

Figure 12. Graduation of Australian national mortality 1970–72 (males) by formula (2). The observed q_x values are indicated by dots.

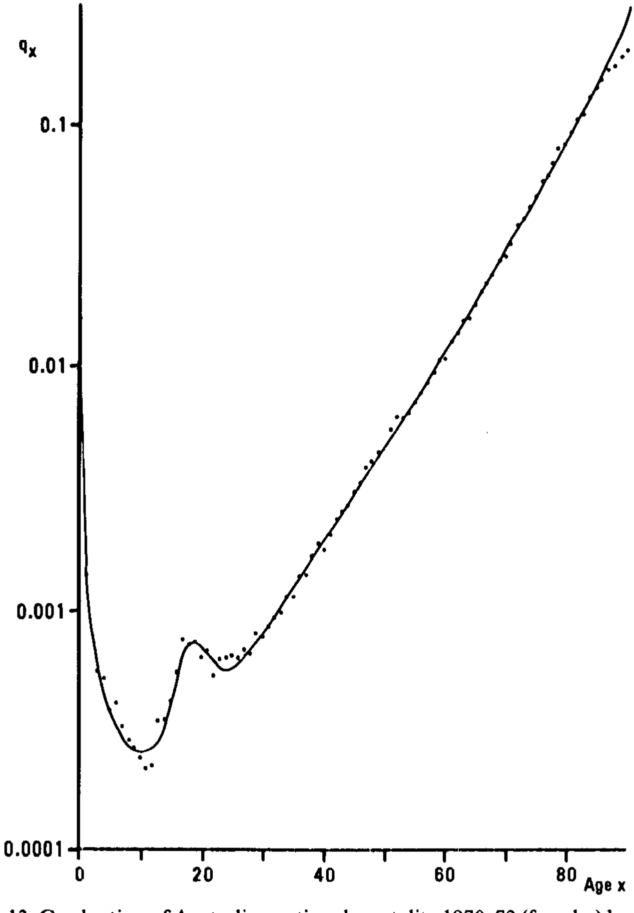

Figure 13. Graduation of Australian national mortality 1970–72 (females) by formula (2). The observed q_x values are indicated by dots.

Table 3. *Parameters for model (3). Australian national mortality 1946–48, 1960–62, and 1970–72.*

Parameter	Males 1946–48	Males 1960–62	Males 1970–72	Females 1946–48	Females 1960–62	Females 1970–72
A	·00334	·00180	·00159	·00308	·00192	·00144
B	·0151	·0068	·0071	·0782	·0707	·0498
C	·1174	·1034	·1013	·1713	·1654	·1494
D	·00102	·00114	·00165	·00059	·00021	·00037
E	7·52	10·72	15·52	3·42	22·37	24·25
F	20·69	20·80	20·10	31·73	19·16	18·24
G	·0000198	·0000171	·0000136	·0002152	·0001261	·0000941
(x_o)	(103·1)	(103·1)	(102·0)	(100·1)	(102·1)	(103·4)
H	1·3222	1·3138	1·3495	1·0151	1·0225	1·0351
K	·789	·797	·783	1·375	1·297	1·206

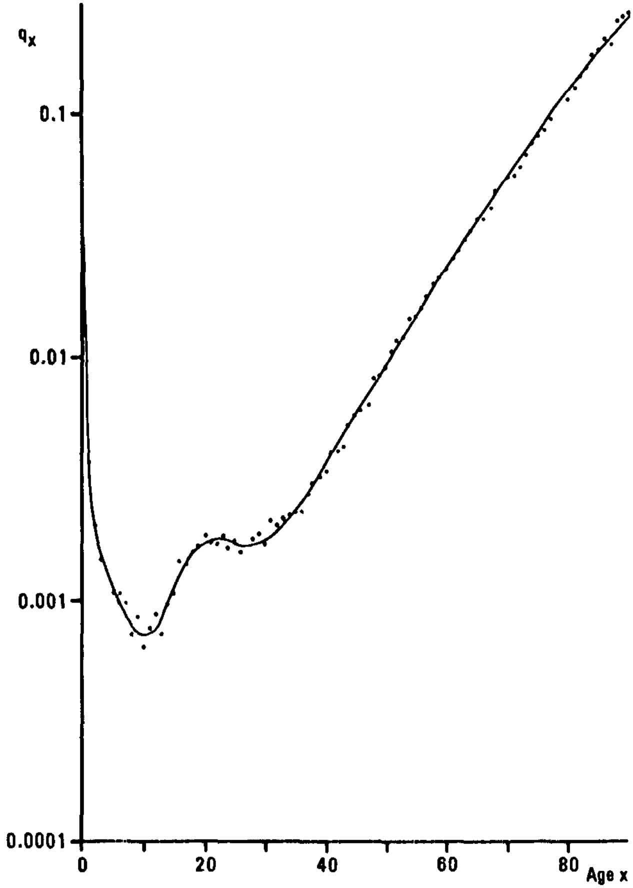

Figure 14. Graduation of Australian national mortality 1946–48 (males) by formula (3). The observed q_x values are indicated by dots.

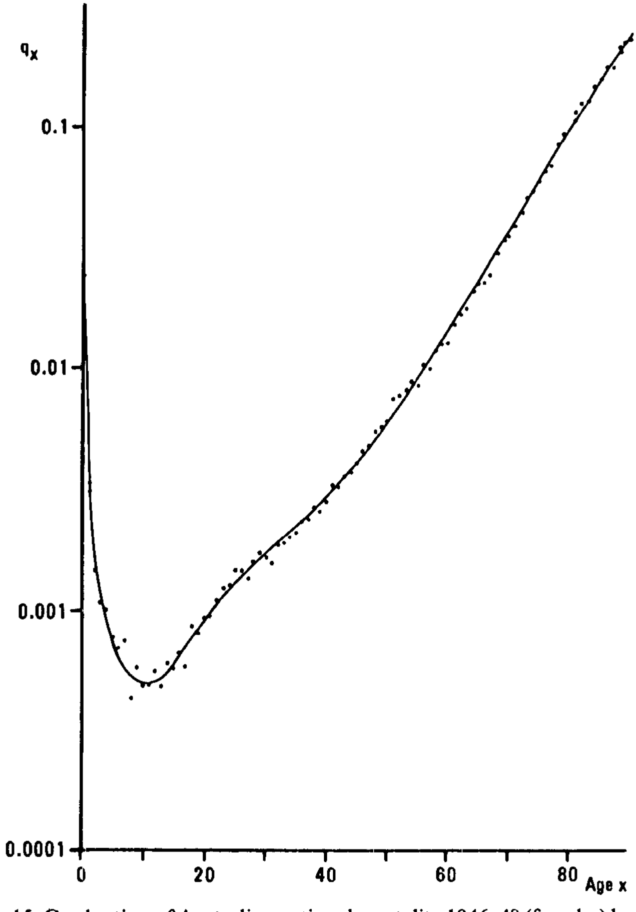

Figure 15. Graduation of Australian national mortality 1946–48 (females) by formula (3). The observed q_x values are indicated by dots.

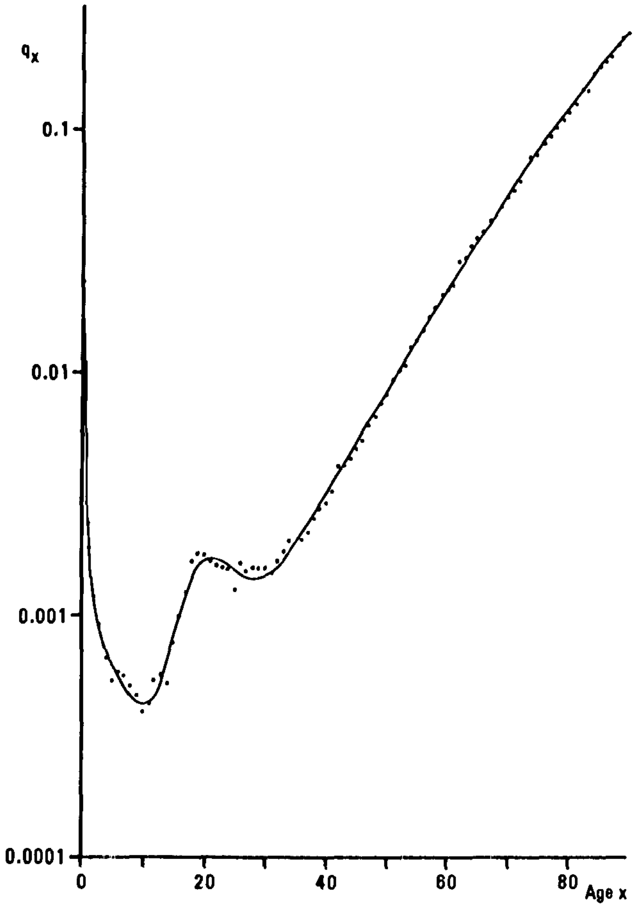

Figure 16. Graduation of Australian national mortality 1960–62 (males) by formula (3). The observed q_x values are indicated by dots.

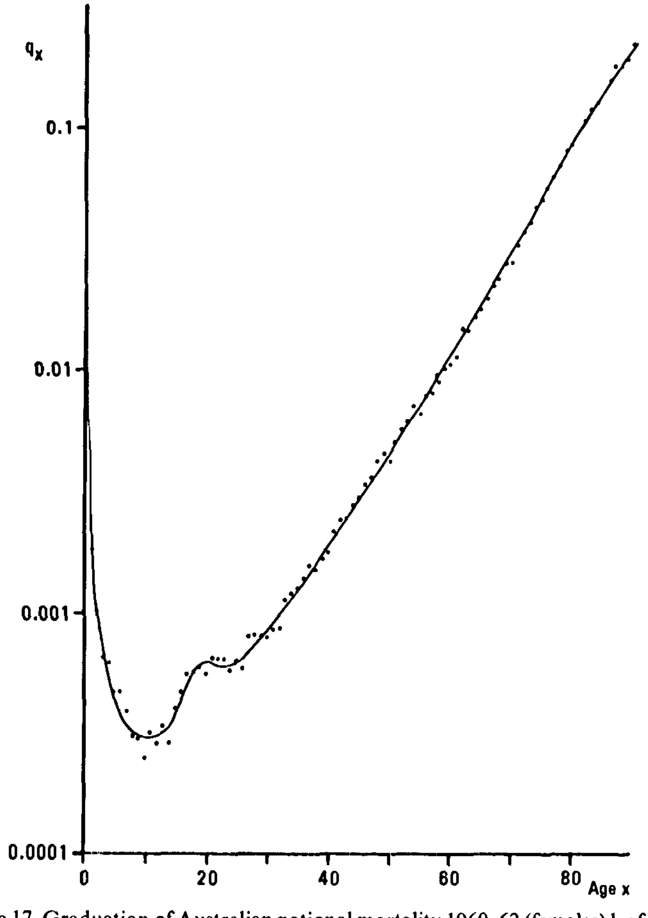

Figure 17. Graduation of Australian national mortality 1960–62 (females) by formula (3). The observed q_x values are indicated by dots.

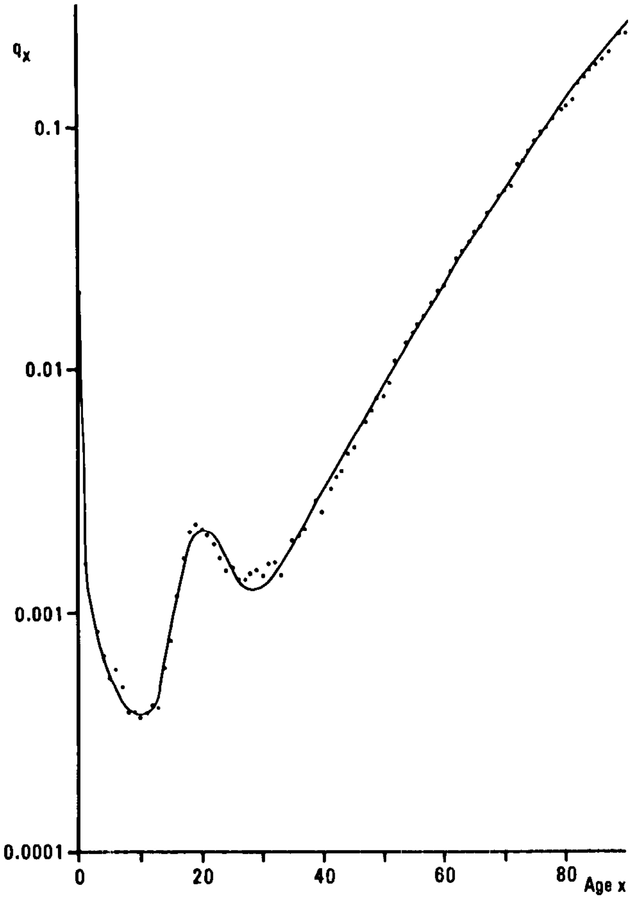

Figure 18. Graduation of Australian national mortality 1970–72 (males) by formula (3). The observed q_x values are indicated by dots.

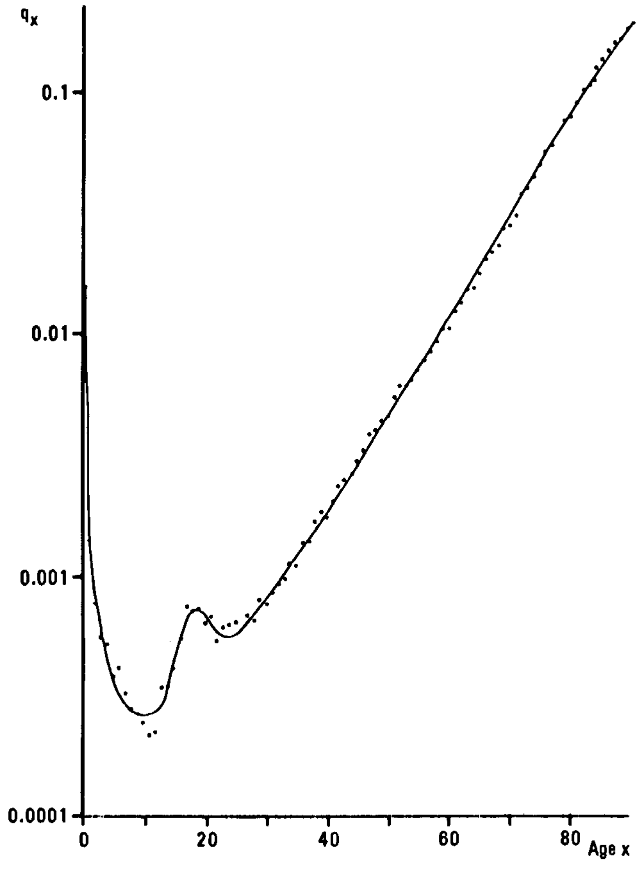

Figure 19. Graduation of Australian national mortality 1970–72 (females) by formula (3). The observed q_x values are indicated by dots.

For model (2) the female base level mortality parameter G shows a continuous decline similar to that of the males.

4.6 The values of G and H for model (3) are very much affected by the direction and extent of curvature (K) and it is not clear that they still have demographic meaning, although x^K can be regarded as a transformation of the age scale. Models (2) and (3) both reduce to (1a) when $K=1$. Increased values of K denote increased curvature downward; decreased K increased curvature upward. For males both models indicate curvature downward, whereas for females the converse is true.

5. CONCLUDING REMARKS

5.1 In this paper we suggest a 'law of mortality' which we believe will describe the age pattern of mortality adequately for a wide variety of experiences. Although the 'law' may not always give a fit close enough for certain actuarial purposes it does reproduce the three distinct features of mortality: that of a child adapting to its new environment, the ageing of the body, and a superimposed accident mortality, and it allows mortality comparisons by age and sex both among countries and within the same country over time. The curve is continuous and applicable over the entire age range; it allows the mortality rate (q_x) to take values between zero and unity only, and has relatively few parameters all of which have demographic interpretation and together fully describe the age pattern of mortality.

5.2 The curve gives an adequate representation of post-war Australian mortality for males and females, and preliminary studies with other markedly different experiences (e.g. the Coale–Demeny model life tables[5]) indicate its wide applicability. The curve itself might possibly be used as a base for a model life table system.

5.3 Each of the three terms in (1) can be put in the same general form:

$$Ae^{-B|f(x)-C|^D},$$

and one might conjecture that a more general 'law' of mortality might be expressed in the form

$$q_x/p_x = \sum_{i=1}^{n} A_i e^{-B_i|f_i(x)-C_i|^{D_i}}$$

of which our curve is a special case (with n = 3). There is, however, a danger of over-parameterization of the curve. Furthermore, if such a generalization is adopted with n > 3 there is some difficulty interpreting the various parameters.

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APPENDIX

Table A1: Graduation of Australian national mortality data (Males) 1946–48, 1960–62, and 1970–72 by formulae (1), (2), and (3)

	1946–48				1960–62				1970–72			
Age x	Observed $q_x^o \times 10^5$	Fitted $q_x \times 10^5$ by formula (1)	(2)	(3)	Observed $q_x^o \times 10^5$	Fitted $q_x \times 10^5$ by formula (1)	(2)	(3)	Observed $q_x^o \times 10^5$	Fitted $q_x \times 10^5$ by formula (1)	(2)	(3)
0	3066	3067	3067	3067	2295	2295	2295	2296	2015	2013	2014	2014
1	362	343	341	334	187	189	186	181	155	167	164	160
2	196	209	208	208	118	115	115	115	97	101	101	101
3	144	154	154	156	90	86	86	87	82	75	76	77
4	128	124	125	126	66	70	70	72	66	61	62	63
5	104	106	106	107	53	60	61	62	52	53	54	55
6	104	93	93	94	58	54	54	55	57	47	48	49
7	95	84	84	85	55	49	50	50	48	44	44	44
8	71	78	78	77	50	47	47	47	38	41	41	41
9	83	73	73	73	46	45	44	44	38	40	39	39
10	62	71	71	70	39	44	43	43	37	39	38	38
11	75	70	70	70	42	44	43	43	38	39	38	37
12	85	73	74	74	53	46	45	45	41	40	40	39
13	71	81	81	82	56	51	51	51	40	45	45	45
14	93	92	92	93	51	61	61	62	56	57	58	58
15	103	107	107	107	76	76	77	78	75	80	80	81
16	140	124	123	123	97	97	97	97	114	112	112	112
17	138	140	139	138	121	119	118	118	162	149	147	146
18	155	154	153	152	162	141	138	137	210	184	180	178
19	163	165	164	162	174	157	154	153	221	208	204	201
20	182	172	171	170	171	168	165	163	214	218	214	212
21	169	175	175	174	167	172	170	168	204	213	212	210
22	168	175	175	175	159	170	169	169	189	199	200	199
23	179	173	174	175	158	164	165	165	162	179	182	183
24	160	170	171	173	151	156	158	160	147	159	163	165
25	172	167	168	170	126	149	152	153	149	142	146	148
26	155	165	166	168	161	142	145	147	134	130	133	135
27	166	164	166	167	149	139	141	143	133	124	125	126
28	173	166	166	168	154	137	139	140	140	122	121	121
29	183	169	169	170	153	139	139	140	148	124	122	121
30	168	175	174	174	154	144	143	143	139	129	126	125
31	209	183	181	180	149	151	148	148	153	138	133	131
32	199	193	191	189	163	160	157	156	156	148	143	141
33	213	205	203	200	180	172	167	166	140	161	155	153
34	220	220	217	213	199	186	180	179	168	176	169	167
35	222	237	233	229	195	202	196	194	195	193	185	184
36	224	256	252	248	199	220	213	211	200	211	204	203
37	265	278	273	269	216	240	233	232	216	232	224	224
38	299	302	297	293	248	263	256	254	254	254	247	247
39	317	329	324	320	271	288	281	280	280	280	273	274

Table A1 (Continued)

Age	Observed	1946–48 Fitted $q_x \times 10^5$ by formula			Observed	1960–62 Fitted $q_x \times 10^5$ by formula			Observed	1970–72 Fitted $q_x \times 10^5$ by formula		
x	$q_x^o \times 10^5$	(1)	(2)	(3)	$q_x^o \times 10^5$	(1)	(2)	(3)	$q_x^o \times 10^5$	(1)	(2)	(3)
40	334	359	354	351	288	315	308	308	256	307	301	303
41	402	392	387	385	319	346	339	340	311	338	332	335
42	407	428	424	422	408	379	373	375	352	371	367	370
43	420	468	464	464	408	416	411	413	377	409	405	410
44	510	512	509	509	433	457	452	456	440	449	447	453
45	574	560	558	560	481	502	498	502	465	494	494	501
46	590	613	612	615	517	551	549	554	565	544	545	554
47	620	671	671	676	594	605	605	611	596	599	602	612
48	811	735	736	743	652	664	666	673	668	659	666	675
49	829	805	807	816	733	729	734	741	753	725	735	746
50	894	881	886	896	804	800	808	817	762	798	812	823
51	1043	965	972	984	915	879	890	899	866	878	897	908
52	1165	1057	1066	1081	988	965	981	990	1075	966	990	1001
53	1173	1158	1170	1186	1044	1060	1080	1089	1120	1063	1093	1103
54	1409	1268	1283	1301	1261	1163	1189	1198	1271	1170	1206	1216
55	1454	1388	1408	1427	1321	1277	1309	1317	1367	1287	1331	1339
56	1582	1520	1544	1565	1476	1402	1441	1447	1486	1416	1468	1474
57	1721	1665	1693	1715	1666	1540	1585	1590	1605	1558	1619	1622
58	1954	1823	1856	1879	1799	1690	1744	1746	1864	1713	1785	1784
59	2059	1996	2034	2057	2024	1855	1917	1916	2043	1884	1967	1961
60	2276	2185	2229	2252	2117	2035	2108	2102	2141	2072	2167	2155
61	2535	2391	2442	2464	2223	2233	2316	2305	2430	2278	2386	2366
62	2718	2617	2674	2694	2796	2450	2543	2527	2749	2504	2626	2597
63	2954	2863	2927	2945	2885	2687	2791	2768	2953	2752	2888	2849
64	3212	3132	3203	3217	3225	2946	3062	3031	3275	3024	3173	3124
65	3637	3426	3503	3513	3486	3230	3357	3317	3590	3322	3485	3424
66	3661	3746	3830	3834	3726	3540	3678	3629	3732	3648	3824	3750
67	4069	4094	4184	4182	4163	3879	4027	3967	4298	4004	4193	4104
68	4728	4474	4569	4559	4383	4249	4405	4334	4517	4394	4593	4489
69	4981	4887	4985	4966	4791	4652	4815	4733	5068	4820	5026	4908
70	5400	5336	5436	5407	5188	5092	5257	5165	5306	5286	5494	5361
71	5525	5824	5922	5883	5429	5570	5735	5633	5566	5793	5998	5853
72	6025	6353	6447	6397	6051	6091	6249	6139	6844	6346	6540	6384
73	6797	6928	7012	6951	6787	6657	6800	6685	7047	6948	7121	6959
74	7483	7550	7618	7547	7561	7272	7391	7275	7614	7602	7743	7579

Table A1 (Continued)

Age	1946–48 Observed	1946–48 Fitted $q_x \times 10^5$ by formula			1960–62 Observed	1960–62 Fitted $q_x \times 10^5$ by formula			1970–72 Observed	1970–72 Fitted $q_x \times 10^5$ by formula		
x	$q_x^0 \times 10^5$	(1)	(2)	(3)	$q_x^0 \times 10^5$	(1)	(2)	(3)	$q_x^0 \times 10^5$	(1)	(2)	(3)
75	8135	8223	8269	8188	7601	7939	8023	7911	8501	8312	8405	8248
76	8567	8950	8965	8876	8492	8661	8695	8595	9318	9082	9109	8968
77	9412	9735	9707	9613	9103	9442	9410	9330	9643	9916	9854	9742
78	10899	10581	10497	10403	10027	10286	10166	10119	10635	10818	10640	10573
79	11461	11491	11336	11248	10808	11196	10964	10965	11227	11791	11466	11462
80	11542	12468	12225	12149	11323	12176	11802	11870	11944	12838	12330	12414
81	12977	13516	13162	13109	12286	13228	12681	12836	12510	13964	13231	13430
82	14161	14637	14148	14130	14290	14357	13597	13865	14691	15171	14165	14512
83	15344	15835	15182	15214	14149	15564	14549	14961	15651	16463	15130	15662
84	17467	17110	16262	16363	16681	16854	15533	16123	16266	17842	16122	16883
85	18413	18466	17387	17577	17370	18227	16548	17355	17579	19310	17136	18174
86	20437	19903	18553	18857	18868	19685	17587	18657	18679	20868	18169	19538
87	19543	21424	19758	20205	19611	21230	18648	20029	19697	22516	19214	20974
88	24303	23026	20997	21619	21789	22861	19725	21472	22475	24255	20267	22482
89	25582	24711	22266	23101	23416	24579	20814	22986	23410	26082	21322	24061
90	26469	26477	23561	24648	24266	26382	21909	24569	23488	27997	22374	25711
91	29344	28322	24876	26260	25539	28267	23005	26220	27895	29995	23417	27429
92	31376	30242	26205	27934	31807	30232	24096	27938	30766	32072	24447	29212
93	31296	32234	27543	29668	30618	32272	25178	29718	29060	34223	25457	31057
94	35815	34293	28883	31458	30333	34382	26244	31559	30146	36441	26445	32959
95	31741	36413	30220	33301	28816	36556	27291	33455	31195	38718	27404	34915
96	30952	38587	31546	35193	42527	38785	28313	35401	35218	41045	28333	36918
97	33784	40807	32858	37127	40181	41063	29307	37393	30769	43413	29227	38962
98	38565	43065	34148	39100	39316	43379	30269	39425	26316	45811	30084	41042
99	34783	45353	35412	41105	36596	45725	31197	41489	32641	48229	30903	43149

Table A2: Graduation of Australian national mortality data (Females) 1946–48, 1960–62, and 1970–72 by formulae (1), (2), and (3)

	1946–48				1960–62				1970–72			
		Fitted $q_x \times 10^5$ by formula				Fitted $q_x \times 10^5$ by formula				Fitted $q_x \times 10^5$ by formula		
Age x	Observed $q_x^o \times 10^5$	(1)	(2)	(3)	Observed $q_x^o \times 10^5$	(1)	(2)	(3)	Observed $q_x^o \times 10^5$	(1)	(2)	(3)
0	2402	2406	2408	2405	1786	1786	1784	1778	1542	1541	1539	1536
1	331	286	287	308	179	177	182	192	137	139	143	147
2	146	166	164	165	100	100	100	99	75	79	80	79
3	107	118	116	113	63	70	70	67	55	57	56	55
4	100	92	91	88	60	55	54	52	51	45	44	43
5	77	75	76	73	46	46	45	43	38	38	37	36
6	69	65	66	64	46	39	39	38	41	33	32	32
7	74	57	59	58	38	35	35	34	32	30	29	29
8	43	53	54	54	30	32	32	32	28	28	27	28
9	58	50	51	51	30	30	30	31	27	26	26	27
10	48	48	49	50	25	29	29	30	24	25	26	26
11	49	49	48	50	31	28	29	30	21	25	25	26
12	55	50	49	50	28	29	29	30	22	26	26	27
13	48	52	50	52	33	31	30	31	34	29	29	29
14	59	56	53	55	28	34	33	33	34	35	35	34
15	57	60	57	59	39	38	38	38	41	44	44	43
16	66	64	63	63	46	43	44	43	54	53	54	54
17	58	70	69	69	54	48	50	50	73	62	64	64
18	85	75	76	75	55	53	55	56	71	68	70	70
19	80	82	83	82	58	57	59	59	72	69	70	71
20	92	88	91	89	54	59	60	61	62	68	68	68
21	93	95	98	96	63	61	61	60	66	65	63	63
22	109	102	106	104	63	62	60	59	52	61	59	58
23	121	110	114	112	63	62	60	59	60	58	56	56
24	123	117	122	120	56	63	60	59	61	56	55	55
25	145	125	130	128	61	63	61	60	63	56	56	56
26	144	133	137	136	57	64	63	63	62	57	58	59
27	135	142	145	144	77	66	66	66	67	60	62	62
28	158	150	152	152	79	69	69	70	64	63	67	67
29	170	159	159	161	79	72	74	75	78	68	72	72
30	161	168	167	169	78	77	80	81	76	74	78	78
31	158	178	175	178	83	82	87	87	84	80	85	84
32	183	188	184	187	84	89	94	94	91	87	92	91
33	189	198	193	196	111	96	102	102	96	96	100	99
34	200	209	202	206	118	104	111	111	111	105	110	108
35	206	221	213	217	125	114	121	120	110	115	119	118
36	232	233	225	228	138	124	132	130	134	125	130	128
37	237	246	238	241	155	136	144	142	137	138	142	140
38	266	261	252	255	150	149	157	154	163	151	155	152
39	255	276	268	270	165	163	171	168	182	166	170	166

Table A2 (Continued)

Age x	1946–48 Observed $q_x^o \times 10^5$	1946–48 Fitted $q_x \times 10^5$ by formula (1)	(2)	(3)	1960–62 Observed $q_x^o \times 10^5$	1960–62 Fitted $q_x \times 10^5$ by formula (1)	(2)	(3)	1970–72 Observed $q_x^o \times 10^5$	1970–72 Fitted $q_x \times 10^5$ by formula (1)	(2)	(3)
40	280	293	285	286	176	179	187	182	172	182	185	181
41	325	311	304	304	212	197	204	199	198	199	203	198
42	320	331	326	324	239	216	223	217	231	219	221	216
43	357	353	350	347	241	237	244	237	248	240	242	237
44	371	377	376	371	271	261	266	258	262	264	265	259
45	404	403	405	398	292	287	291	282	296	290	289	283
46	457	432	437	429	331	315	318	309	326	319	317	310
47	474	464	473	462	353	347	348	338	376	350	346	340
48	544	500	512	499	414	381	381	370	392	385	379	372
49	551	539	555	540	448	419	416	405	432	422	415	408
50	597	583	602	585	415	461	455	444	449	464	454	448
51	748	631	654	635	494	507	498	487	542	510	497	491
52	767	685	711	690	559	558	545	534	606	561	544	539
53	824	744	774	751	604	613	597	586	602	616	595	592
54	879	810	842	819	697	675	654	644	629	677	652	651
55	847	883	918	894	654	742	716	707	696	744	714	715
56	1018	965	1001	977	777	816	784	777	775	817	782	786
57	992	1055	1092	1069	793	898	859	855	840	898	857	864
58	1190	1156	1192	1170	885	987	941	940	925	987	940	950
59	1279	1267	1302	1282	995	1085	1032	1035	1027	1084	1030	1045
60	1275	1391	1423	1406	1031	1194	1131	1139	1033	1191	1130	1149
61	1524	1528	1555	1543	1116	1312	1240	1255	1221	1308	1239	1265
62	1675	1680	1701	1695	1466	1443	1361	1383	1337	1437	1359	1392
63	1778	1849	1861	1863	1432	1586	1494	1524	1506	1578	1492	1533
64	2050	2037	2037	2048	1612	1743	1640	1681	1519	1732	1638	1688
65	2219	2244	2231	2254	1746	1916	1801	1854	1717	1902	1799	1859
66	2256	2474	2445	2481	1960	2105	1980	2045	1800	2088	1977	2047
67	2420	2729	2681	2732	2214	2313	2177	2256	2142	2292	2173	2255
68	2973	3010	2941	3010	2389	2541	2395	2490	2307	2515	2390	2485
69	3357	3322	3228	3317	2707	2790	2637	2749	2710	2759	2631	2738
70	3513	3666	3546	3656	2738	3063	2905	3035	2785	3026	2897	3016
71	3903	4047	3898	4031	3257	3362	3204	3351	3081	3318	3193	3323
72	4401	4466	4288	4445	3697	3689	3536	3701	3805	3638	3521	3661
73	5010	4929	4722	4903	4075	4047	3906	4087	3977	3986	3887	4033
74	5395	5438	5204	5407	4714	4437	4320	4513	4465	4367	4295	4443

Table A2 (Continued)

Age	1946–48				1960–62				1970–72			
	Observed	Fitted $q_x \times 10^5$ by formula			Observed	Fitted $q_x \times 10^5$ by formula			Observed	Fitted $q_x \times 10^5$ by formula		
x	$q_x^o \times 10^5$	(1)	(2)	(3)	$q_x^o \times 10^5$	(1)	(2)	(3)	$q_x^o \times 10^5$	(1)	(2)	(3)
75	5970	5998	5743	5964	4959	4864	4784	4983	4942	4782	4751	4893
76	6594	6613	6344	6577	5480	5329	5305	5502	5606	5235	5261	5387
77	6921	7289	7018	7251	6161	5836	5892	6074	5876	5728	5835	5931
78	8526	8029	7775	7993	6921	6388	6555	6703	6725	6265	6481	6526
79	9318	8838	8628	8808	7947	6988	7308	7395	7686	6848	7210	7179
80	9724	9721	9793	9702	8240	7640	8166	8155	7981	7480	8038	7893
81	11395	10684	10689	10681	9146	8347	9148	8990	9017	8167	8980	8675
82	12317	11732	11940	11751	10557	9114	10279	9903	10287	8910	10059	9527
83	12768	12868	13374	12918	11906	9943	11590	10903	10943	9714	11299	10456
84	14791	14099	15029	14189	12548	10839	13121	11993	12767	10581	12736	11468
85	15635	15427	16952	15568	13651	11804	14924	13181	13771	11517	14410	12565
86	17675	16857	19205	17063	15126	12844	17069	14473	15018	12523	16380	13755
87	17553	18392	21868	18676	17368	13961	19654	15872	16053	13604	18719	15041
88	21438	20035	25053	20412	17286	15157	22813	17385	16847	14763	21530	16428
89	22509	21785	28913	22273	18556	16437	26747	19015	18433	16002	24957	17920
90	23506	23645	33667	24261	21553	17802	31755	20765	19705	17323	29208	19519
91	26804	25612	39642	26375	22315	19255	38319	22638	21644	18730	34596	21228
92	27175	27684	47344	38614	26628	20796	47252	24634	23067	20223	41615	23048
93	27604	29858	57603	30972	26547	22426	60061	26752	24393	21803	51092	24979
94	30317	32127	71878	33444	27246	24145	79863	28990	27475	23470	64529	27019
95	29580	34484	93005	36021	30677	25952	*	31342	27653	25224	84958	29166
96	31429	36921	*	38691	29966	27844	*	33802	30970	27062	*	31416
97	32323	39427	*	41443	33655	29818	*	36361	33973	28982	*	33761
98	31707	41991	*	44259	33613	31871	*	39008	30588	30981	*	36195
99	50000	44599	*	47124	34146	33997	*	41730	27648	33053	*	38706

* outside the unit interval (0,1)