Moral-Hazard Premium

Takashi MISUMI Hisashi NAKAMURA
Koichiro TAKAOKA
Hitotsubashi University

July 2014
@ APRIA Moscow

Motivation

- It is well known that moral hazard (MH) is one of the most serious problems in finance.
- Huge literature on MH in corporate finance (e.g., Tirole (2006, § 3.2)).
  - Typically, consider the case that the prob of success of a firm's project depends on the firm manager's hidden effort.
  - An investor then needs to provide the manager with an incentive to avoid his opportunistic misbehavior.

  ⇒ The MH problem distorts optimal risk sharing and allocation.
- Such micro effects are not all the distortion causes in the financial world.

⇒ The distortion should also influence the valuation of the firm and all other financial assets in macro markets
  - Via two channels:
    - Distorted allocation
    - Distorted pricing kernel (i.e., marginal utility of the investor)

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Motivation (cont.)

- However, there are few studies of the valuation of MH in asset pricing (AP) and financial engineering (FE).
  - How much return does the investor demands as compensation for a loss caused by MH?
  - E.g., fixed-income investment, the term structure of interest rates, corporate risk management, and profit loadings in insurance.
- A notable exception that studies AP in the presence of MH: Ou-Yang (2005)
  - On the assumption of:
    - Exponential utility function
    - → Needs more general utility functions, such as the one of a power type.
    - Exogenous constant riskless rate
    - → Irrelevant in practice.

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Purpose of this paper

- Provides an explicit asset-pricing formula in general equilibrium (GE) under MH on the assumption of:
  - CRRA/log utility.
  - Endogenous riskless rate.
- Makes clear the structural effect of MH on macro markets.
  - Prices any kinds of securities in the contingent-claims approach
  - Still, not an insurance model, which is my next research topic.

Skip to Results

Key characteristics of the model

- Incorporates MH into a GE exchange economy.
  - A representative investor and a representative firm (i.e., firm manager) over $[0, T]$.
    - Standard GE exchange models: Lucas (1978), Breeden (1979), Cox et al. (1985), Dana and Jeanblanc (2007, Ch.7).
    - Focus on the macro effect of MH, NOT the micro one.
  - The firm produces a single non-storable cons good over time.
    - No productive resources are utilized → Endowment.
    - Can control the probability measure ex post with effort costs.
  - ⇒ The endowment process is subject to MH.
    - MH: The firm's ex-post, costly, strategic measure change.
- Investigates GE in the presence of MH.
  - The investor controls ex-post optimally consumption/wealth while having access to financial markets
    - Under regular risk B and rare-event risk M.
  - Also, designs ex-ante optimally an incentive contract for the manager under MH.

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Results

- Provides an explicit equilibrium state price under MH
  - State price Π (or, pricing kernel, stochastic discount factor),

$$\frac{\mathrm{d}\Pi(t)}{\Pi(t_-)} = -r(t)\,\mathrm{d}t - \eta(t)^\top\,\mathrm{d}B(t) - \xi(t)^\top\,\mathrm{d}M(t), \quad \Pi(0) = 1.$$

1. Shows how MH distorts r, η, ξ.
   - The market price of diffusive risk η is distorted in the opposite direction of the investor's marginal utility.
   - The market price of jump risk ξ is raised.
   - A positive premium is stipulated on a riskless rate r – called a moral-hazard premium.

   → Implies that the risk-free rate puzzle, explored first by Weil (1989), is more serious in the presence of MH.
2. Unlike in standard corporate-finance models, the markets alleviate the allocation conflict caused by MH.

Skip to Outline of this presentation

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Contributions

- Not a generalization of Ou-Yang (2005), but rather complementary to it.
  - (+) Generalizes the utility function into the power utility one.
  - (+) Endogenizes the riskless rate.
  - (−) Restricts the contract form to being a stationary linear payment rule: $S(t) = sX(t)$.
    - Linear (i.e., proportional to the production) at a constant (i.e, time-independent) rate of change.
    - Mathematically, without any such approximations, hard to solve the stochastic-control problem with state constraints.
- Conjecture: How restrictive is the stationary linear contract?
  - The linearity is NOT, (b/c) the system of the eqn's is linear.
  - The stationarity IS, (b/c) it is the finite-period model.
  - → By perturbing the above results, investigate the effect of non-stationarity.
- Our explicit solutions are a good benchmark for future numerical approximations under more general contract forms.

Go to Future work
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Related literature

Related also to the literature on cont-time optimal contracting in the presence of MH (see e.g. Cvitanić and Zhang (2013)).

- **Similarities**: Our paper and Cvitanić and Zhang (2007) (CZ) both study MH on the assumption of more general utility function than the exponential one in the weak formulation.
  - CZ deal with adverse selection as well.
  - Nakamura and Takaoka (2013) solve the optimal contracting problem in the strong formulation.
- **Departures**: An extension of CZ's optimal-contracting model into an asset-pricing model.
  - Ou-Yang (2005) based on Holmström and Milgrom (1987), Schättler and Sung (1993).
  - Our paper based on CZ.

Other technological departures from CZ

1. Two types of risk: Brownian motions as 'regular risk' and Poisson processes as 'rare-event risk.'
2. The consumption takes place all over $[0, T]$
   - CZ assume that it does only at the terminal point T.
3. The firm controls directly the probability measure.
   - In the spirit of the standard MH literature (e.g., Tirole (2006))
     - Easy to deal with jump risk.
   - In contrast, CZ assume that the agent controls the drift rate.
     - See also Sannikov (2008), Holmström and Milgrom (1987), Schättler and Sung (1993).
   - Effort cost= relative entropy
     - A 'distance' between the reference (i.e., original) measure and the controlled probability measure.
     - See also Hansen and Sargent (2007), Hansen et al. (2006), Sims (2003), Delbaen et al. (2002).

Outline of this presentation

1. Set-up
2. Optimization
3. Market equilibrium
4. Equilibrium asset prices
5. Concluding remarks

Set-up

- 2 risk-averse players:
  - 1 representative investor
  - 1 representative firm (i.e., firm manager)
- Continuous time $[0, T]$, $T > 0$.
- Filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.
  - $\{B_1(t), \cdots, B_n(t)\}_{0 \le t \le T}$: n independent 1-dim stan. $\mathbb{F}$-Brownian motions (BMs).
  - $\{N_1(t), \cdots, N_m(t)\}_{0 \le t \le T}$: m independent Poisson processes w/ its intensity $\lambda_i > 0$ $(i = 1, \cdots, m)$. Indep of the n BMs.
    - Compensated Poisson process $M_i(t) := N_i(t) - \lambda_i t$.
  - $\mathbb{F}$ is generated by the n BMs and the m Poisson processes.
- The firm produces a single consumption good and shares it with the investor over time.

Production

- THE firm produces X characterized by SDEs:

$$\begin{aligned} \mathrm{d}X(t) &= X(t_-)\,\mathrm{d}G(t), \quad X(0) = x > 0 \\ \mathrm{d}G(t) &= \mu^G\,\mathrm{d}t + \sum_{j=1}^{n} \sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^{m} z_i^G\,\mathrm{d}M_i(t), \end{aligned}$$

  where $\mu^G, \sigma_j^G, z_i^G\,\forall\, i, j$ are constants, $\sigma_j^G > 0\,\forall\, j$, $z_i^G > -1\,\forall\, i$, and $z_{i_1}^G \neq z_{i_2}^G$ if $i_1 \neq i_2$.
  - $\{B_j;\, j = 1, \cdots, n\}$ stands for regular risk
  - $\{N_i;\, i = 1, \cdots, m\}$ stands for rare-event (i.e., jump) risk.
  - For each $i = 1, \cdots, m$, z_i^G denotes the size of the jump.
- No productive resources are utilized.
  - The production is an endowment.

Moral hazard

- THE firm can control the probability measure with costly efforts.
  - Changes the probability measure from the original (reference) $\mathbb{P}$ into $\mathbb{Q}$ such that $\mathbb{Q} \ll \mathbb{P}$.
  - Effort cost: separable utility cost, characterized by relative entropy:

$$
\begin{aligned}
H(\mathbb{Q}\,||\,\mathbb{P}) &:= \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\left(\log\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\mathbf{1}_{\{\frac{d\mathbb{Q}}{d\mathbb{P}}>0\}}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\left(\log\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\mathbf{1}_{\{\frac{d\mathbb{Q}}{d\mathbb{P}}>0\}}\right] = \mathbb{E}^{\mathbb{Q}}\left[\log\frac{d\mathbb{Q}}{d\mathbb{P}}\right].
\end{aligned}
$$

  - Assume that $H(\mathbb{Q}\,||\,\mathbb{P}) < \infty$.
- Definition of Moral hazard:
  The firm's ex-post, costly, strategic measure change.

Go to Note on measure change

Consumption good

- The firm and the investor consume the cons good X over $[0, T]$:
  - The firm $S = \{S(t)\}_{0 \leq t \leq T} \in \mathcal{H}_+$
  - The investor $C = \{C(t)\}_{0 \leq t \leq T} \in \mathcal{H}_+$

  with $C(T) = 0, S(T) = 0$.
- Shared according to terms of a contract – call S a contract or a payment rule.
  - S: the firm's share and consumption.
  - C: the investor's consumption out of her share $X - S$.

  — The details of the contract space will be defined below.

Financial markets

- Financial markets are accessible only to the investor, not to the firm.
- Riskless asset: $\frac{\mathrm{d}P_0(t)}{P_0(t)} = r(t)\,\mathrm{d}t$
- d risky assets: its excess return

$$\mathrm{d}R(t) = \mu^R\,\mathrm{d}t + \sum_{j=1}^{n} \sigma_j^R\,\mathrm{d}B_j(t) + \sum_{i=1}^{m} z_i^R\,\mathrm{d}M_i(t)$$

- The investor's initial funds $W(t) = w_0 = 0$
- Wealth process W is characterized by the following SDE:

$$\begin{aligned}\mathrm{d}W(t) &= W(t_-)r(t)\,\mathrm{d}t + W(t_-)\beta(t)^\top\,\mathrm{d}R(t) \\ &\quad + \Big(X(t) - C(t) - S(t)\Big)\,\mathrm{d}t, \qquad W(0) = w_0.\end{aligned}$$

Set of controls

Definition

Define $\mathcal{A}$ as the set of the control triples $(S, C, \beta) \in \mathcal{H}_+ \times \mathcal{H}_+ \times \mathcal{B}$ such that

(i) $0 < S(t) \leq X(t)\ \forall\, [0, T)$ a.s. and $S(T) = 0$,

(ii) S is of the Markovian form $S(t) := \tilde{s}(t, X(t))$ for some deterministic function $\tilde{s}$. In particular, we restrict the contract form as follows: for some constant $s \in (0, 1) \in \mathbf{R}_{++}$, $S(t) = sX(t)\, \forall\, t \in [0, T)$ a.s.. Call this form a stationary linear payment rule.

(iii) $\mathbb{P}[\int_0^T e^{-\delta u} \log S(u)\,\mathrm{d}u > -\infty] > 0$,

(iv) $0 < C(t) \leq X(t)\ \forall\, [0, T)$ a.s. and $C(T) = 0$,

(v) C is of the Markovian form $C(t) := c(t, X(t), W(t))$ for some deterministic function c,

(vi) β is of the Markovian form $\beta(t) := b(t, X(t), W(t))$ for some deterministic function b,

(vii) The wealth process W, which is generated via its evolution, satisfies $\mathbb{E}^{\mathbb{P}}[(W(T))^2] < \infty$.

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Firm's optimization

- The firm's utility

$$V_1 := \sup_{\substack{\mathbb{Q} \ll \mathbb{P} \\ H(\mathbb{Q}||\mathbb{P})<\infty}} U_1(S;\mathbb{Q})$$

where $$U_1(S;\mathbb{Q}) := \mathbb{E}^{\mathbb{Q}}[\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u] - H(\mathbb{Q}\,||\,\mathbb{P}).$$

- Participation constraint (PC,hereafter): $V_1 \geq \rho \in \mathbf{R}$.

Firm's optimization (cont.)

Lemma

For S,

$$V_1 = \log \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}\right]$$

with the maximizer $\mathbb{Q}^*$ *characterized by:*

$$\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} = e^{-V_1} e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}.$$

$\Rightarrow$ This works as an incentive constraint for the firm in the investor's optimization.

Go to Proof

Investor's optimization

$$
\begin{aligned}
V_2(0) &= \sup_{(S,C,\beta)\in\mathcal{A}} U_2\left(C, W(T)\,;\,\mathbb{Q}^*\right) && (*)\\
\text{s.t. } \mathrm{d}W(t) &= W(t_-)r(t)\,\mathrm{d}t + W(t_-)\beta(t)^\top\,\mathrm{d}R(t) + \Big(X(t) - C(t) - S(t)\Big)\,\mathrm{d}t,\\
\mathrm{d}R(t) &= \mu^R\,\mathrm{d}t + \sum_{j=1}^{n}\sigma_j^R\,\mathrm{d}B_j(t) + \sum_{i=1}^{m} z_i^R\,\mathrm{d}M_i(t),\ W(0) = w_0,\\
\mathrm{d}X(t) &= X(t_-)\Big(\mu^G\,\mathrm{d}t + \sum_{j=1}^{n}\sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^{m} z_i^G\,\mathrm{d}M_i(t)\Big),\ X(0) = x_0,\\
V_1 &= \log\mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}\right] \geq \rho, && \text{[Participation constraint]}\\
\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} &= e^{-V_1} e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u} && \text{[Incentive constraint]}
\end{aligned}
$$

where $U_2\left(C, W(T)\,;\,\mathbb{Q}^*\right) := \mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T e^{-\delta u}\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T} W(T)\right].$

Lagrangian

$$
\begin{aligned}
&\sup_{(S,C,\beta)\in\mathcal{A}} \mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T e^{-\delta u}\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T}W(T)+\chi\right] \\
= &\sup_{(S,C,\beta)\in\mathcal{A}} e^{-V_1}\mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a\log S(u)\,\mathrm{d}u}\left(\int_0^T e^{-\delta u}\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T}W(T)+\chi\right)\right]
\end{aligned}
$$

s.t.

$$
\begin{aligned}
\mathrm{d}W(t) &= W(t_-)r(t)\,\mathrm{d}t + W(t_-)\beta(t)^\top\,\mathrm{d}R(t) + \Big(X(t)-C(t)-S(t)\Big)\,\mathrm{d}t, \\
\mathrm{d}R(t) &= \mu^R\,\mathrm{d}t + \sum_{j=1}^n \sigma_j^R\,\mathrm{d}B_j(t) + \sum_{i=1}^m z_i^R\,\mathrm{d}M_i(t), \quad W(0)=w_0, \\
\mathrm{d}X(t) &= X(t_-)\Big(\mu^G\,\mathrm{d}t + \sum_{j=1}^n \sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^m z_i^G\,\mathrm{d}M_i(t)\Big), \quad X(0)=x_0
\end{aligned}
$$

Market equilibrium: Definition

Definition

A control triple $(S, C, \beta) \in \mathcal{A}$ is said to be in market equilibrium if the following conditions are satisfied:

(i) The control triple $(S, C, \beta) \in \mathcal{A}$ is optimal for $(*)$,

(ii) For all t, $C(t) = X(t) - S(t)$,

(iii) For all t, all the elements of $\beta(t) = 0$,

Stationary linear payment rule $S(t) = sX(t)$

$$\sup_{(S,C,\beta)\in\mathcal{A}} \mathbb{E}^{\mathbb{P}}\left[\frac{e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}}{\mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right]}\left(\begin{array}{c}\int_0^T e^{-\delta u}\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u \\ +e^{-\delta T}W(T)+\chi\end{array}\right)\right]$$

$$= \sup_{(S,C,\beta)\in\mathcal{A}} \left(\begin{array}{l}\mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right]^{-1} \cdot \\ \mathbb{E}^{\mathbb{P}}\left[\begin{array}{l}\int_0^T e^{-\delta u}\mathbb{E}_u^{\mathbb{P}}[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}]\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u \\ +e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\left(e^{-\delta T}W(T)+\chi\right)\end{array}\right]\end{array}\right)$$

s.t.

$$\begin{aligned}
\mathrm{d}W(t) &= W(t_-)r(t)\,\mathrm{d}t + W(t_-)\beta(t)^\top\,\mathrm{d}R(t) + \Big(X(t) - C(t) - S(t)\Big)\,\mathrm{d}t, \\
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\end{aligned}$$

Characterization: Measure change

Define a time-t distortion of $\mathbb{Q}$ caused by X over $[0, T]$ as

$$Y(t) := \mathbb{E}_t^{\mathbb{P}}\left[e^{a\int_0^T e^{-\delta u}\log X(u)\,\mathrm{d}u}\right].$$

Lemma

$$Y(t) = \exp\left(\begin{array}{l} a\left(\begin{array}{l}\int_0^t e^{-\delta u}\log X(u)\,\mathrm{d}u + \frac{e^{-\delta t}-e^{-\delta T}}{\delta}\log X(t) \\ +e^{-\delta t}\left(\mu^G - \frac{1}{2}\sum_{j=1}^n(\sigma_j^G)^2 - \sum_{i=1}^m \lambda_i\right)\frac{1-e^{-\delta(T-t)(1+\delta(T-t))}}{\delta^2}\end{array}\right) \\ +\sum_{j=1}^n \frac{(ae^{-\delta t}\sigma_j^G)^2}{2}\frac{1}{\delta^2}\int_0^{T-t}(e^{-\delta u}-e^{-\delta(T-u)})^2\,\mathrm{d}u \\ +\sum_{i=1}^m \lambda_i(T-t)\left(-1+\frac{e^{-\frac{e^{-\delta(T-t)}}{\delta}\log(1+z_i^G)}}{T-t}\int_0^{T-t}e^{\frac{e^{-\delta u}}{\delta}\log(1+z_i^G)}\,\mathrm{d}u\right)\end{array}\right).$$

The martingale $Y(t)$ satisfies

$$\mathrm{d}Y(t) = Y(t_-)\left\{\begin{array}{l} a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\sum_{j=1}^n \sigma_j^G\,\mathrm{d}B_j(t) \\ +\sum_{i=1}^m\left((1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}}-1\right)\mathrm{d}M_i(t)\end{array}\right\}.$$

Verification theorem

If, for a fixed s,

$$0 = \sup_{C(t),\,\beta(t)} \left(K\, Y(t) \frac{C(t)^{1-\gamma}}{1-\gamma} + \mathcal{L}^{C,\beta} J(t, X(t), Y(t), W(t)) \right),$$

then $V_2(t) = J(t, X(t), Y(t), W(t))$, where

$$\begin{aligned}
&\mathcal{L}^{C,\beta} h(t, X(t), Y(t), W(t)) \\
&:= h_t + \mu^G X(t) h_x + \Big(rW(t) + \beta(t)^\top \mu^R W(t) + \Big((1-s)X(t) - C(t) \Big) \Big) h_w \\
&+ \frac{1}{2} \sum_{j=1}^{n} \left\{ (\sigma_j^G)^2 X(t)^2 h_{xx} + (\beta(t)^\top \sigma_j^R)^2 W(t)^2 h_{ww} + (a \frac{e^{-\delta t} - e^{-\delta T}}{\delta} \sigma_j^G)^2 Y(t)^2 h_{yy} \right\} \\
&+ \sum_{j=1}^{n} \left\{ \begin{array}{l} (\beta(t)^\top \sigma_j^R \sigma_j^G) X(t) W(t) h_{xw} + (a \frac{e^{-\delta t} - e^{-\delta T}}{\delta} (\sigma_j^G)^2) X(t) Y(t) h_{xy} \\ +(a \frac{e^{-\delta t} - e^{-\delta T}}{\delta} \beta(t)^\top \sigma_j^R \sigma_j^G) Y(t) W(t) h_{yw} \end{array} \right\} \\
&+ \sum_{i=1}^{m} \lambda_i \left\{ \begin{array}{l} h(t, (1+z_i^G) X(t), (1+z_i^G)^{a \frac{e^{-\delta t} - e^{-\delta T}}{\delta}} Y(t), (1 + \beta(t)^\top z_i^R) W(t)) \\ - h(t, X(t), Y(t), W(t)) \end{array} \right\},
\end{aligned}$$

and $$K := Y(0)^{-1} = \left(\mathbb{E}^{\mathbb{P}} \left[e^{\int_0^T e^{-\delta u} a \log X(u)\, \mathrm{d}u} \right] \right)^{-1} = e^{-\rho + \frac{(1-e^{-\delta T}) a \log \hat{s}}{\delta}}.$$

Optimality and equilibrium

Optimality for s:
From the participation constraint,

$$s^* = \exp\left\{\frac{\delta}{a(1-e^{-\delta T})}\left(\rho - \log \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\, \mathrm{d}u}\right]\right)\right\}.$$

on the assumption of $\log \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\, \mathrm{d}u}\right] \geq \rho$.
Since $Y(0) = \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\, \mathrm{d}u}\right]$,

$$s^* = \exp\left\{\begin{array}{l} \frac{\delta}{a(1-e^{-\delta T})} \times \\ \left(\rho - \left(\begin{array}{l} a\left(\begin{array}{l} \frac{1-e^{-\delta T}}{\delta}\log x_0 \\ +\left(\mu^G - \frac{1}{2}\sum_{j=1}^n (\sigma_j^G)^2 - \sum_{i=1}^m \lambda_i\right)\frac{1-e^{-\delta T(1+\delta T)}}{\delta^2} \end{array}\right) \\ +\sum_{j=1}^n \frac{(a\sigma_j^G)^2}{2}\frac{1}{\delta^2}\int_0^T (e^{-\delta u} - e^{-\delta T})^2\, \mathrm{d}u \\ +\sum_{i=1}^m \lambda_i T\left(-1 + \frac{e^{-\frac{e^{-\delta T}}{\delta}\log(1+z_i^G)}}{T}\int_0^T e^{\frac{e^{-\delta u}}{\delta}\log(1+z_i^G)}\, \mathrm{d}u\right) \end{array}\right)\right) \end{array}\right\}.$$

Optimality and equilibrium (cont.)

Optimality for C, β:

$$
\begin{aligned}
&[C(t)] \quad K\,Y(t)C(t)^{-\gamma} = J_w > 0,\\
&[\beta(t)] \quad W(t)J_w(\mu^R)^\top + W(t)^2 J_{ww}\beta(t)^\top \sigma^R(\sigma^R)^\top + W(t)X(t)J_{xw}\sigma^G(\sigma^R)^\top\\
&\qquad + W(t)Y(t)J_{wy}\sigma^G(\sigma^R)^\top a\frac{e^{-\delta t} - e^{-\delta T}}{\delta} + W(t)J_w\lambda^\top(z^R)^\top = 0.
\end{aligned}
$$

Therefore, due to the market-clearing conditions,

$$
\begin{aligned}
\mu^R + z^R\lambda &= -\sigma^R(\sigma^G)^\top \frac{X(t)J_{xw} + Y(t)J_{wy}a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}}{J_w},\\
C^*(t) &= (1 - s^*)X(t),\\
S^*(t) &= s^* X(t).
\end{aligned}
$$

The equilibrium wealth $W^*(t) = 0\,\forall\, t$ a.s..

Go to Details of Value function

Equilibrium state price

Proposition

For the optimal contract S^*, an equilibrium state price is

$$\begin{aligned}
\Pi(t) &= \Lambda^*(t) = \mathcal{E}(t) F_c^*(t) \quad \text{for } 0 \le t < T \\
\text{where } F_c^*(t) &:= \widehat{F}_c(t)\Big|_{C(t)=C^*(t)} = e^{-\rho} e^{\frac{1-e^{-\delta T}}{\delta} a \log s^*} Y(t) C^*(t)^{-\gamma}, \\
&= e^{-\rho} e^{\frac{1-e^{-\delta T}}{\delta} a \log s^*} (1-s^*)^{-\gamma} Y(t) X(t)^{-\gamma}, \\
\mathcal{E}(t) &:= e^{-\delta t}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\mathrm{d}\Pi(t)}{\Pi(t_-)} &= -r(t)\,\mathrm{d}t - \eta(t)^\top \mathrm{d}B(t) - \xi(t)^\top \mathrm{d}M(t) \\
= \frac{\mathrm{d}\Lambda^*(t)}{\Lambda^*(t_-)} &= -\delta\,\mathrm{d}t + \frac{\mathrm{d}F_c^*(t)}{F_c^*(t_-)}, \qquad \text{with} \quad \Pi(0) = \Lambda(0) = 1.
\end{aligned}$$

Go to Derivation of Equilibrium state price

Eqm riskless rate and MPRs: No moral hazard

- Under the stationary linear payment rule, $\frac{\mathrm{d}\left(C(t)^{-\gamma}\right)}{C(t_-)^{-\gamma}} = \frac{\mathrm{d}\left(X(t)^{-\gamma}\right)}{X(t_-)^{-\gamma}}$:

$$
\begin{aligned}
r^s &= \delta + \gamma\mu^G - \frac{\gamma(\gamma+1)}{2}\sum_{j=1}^{n}(\sigma_j^G)^2 + \sum_{i=1}^{m}\left\{1-(1+z_i^G)^{-\gamma}-\gamma z_i^G\right\}\lambda_i, \\
\eta_j^s &= \gamma\sigma_j^G \quad \text{for } j = 1,\cdots,n, \\
\xi_i^s &= 1-(1+z_i^G)^{-\gamma} \quad \text{for } i = 1,\cdots,m.
\end{aligned}
$$

- Optimal sharing ratio s^{**}:

$$s^* \leq s^{**}$$

(b/c) due to binding PCs,

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u\right] \leq \log \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right].$$

– Note: This inequality may look counter-intuitive: Smaller $\frac{\mathrm{d}S}{\mathrm{d}X}$ under MH
  - The markets alleviate the allocation conflict caused by MH.
  - In particular, the whole effect of MH is absorbed in the markets in this model, b/c PC binds under the linear contract.

Eqm riskless rate and MPRs: No moral hazard

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$$\begin{aligned} r^s &= \delta + \gamma\mu^G - \frac{\gamma(\gamma+1)}{2}\sum_{j=1}^{n}(\sigma_j^G)^2 + \sum_{i=1}^{m}\left\{1-(1+z_i^G)^{-\gamma} - \gamma z_i^G\right\}\lambda_i, \\ \eta_j^s &= \gamma\sigma_j^G \quad \text{for } j = 1, \cdots, n, \\ \xi_i^s &= 1-(1+z_i^G)^{-\gamma} \quad \text{for } i = 1, \cdots, m. \end{aligned}$$

- Optimal sharing ratio s^{**}:

$$s^* \leq s^{**}$$

(b/c) due to binding PCs,

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T e^{-\delta u} a \log X(u)\, \mathrm{d}u\right] \leq \log \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\, \mathrm{d}u}\right].$$

– Note: This inequality may look counter-intuitive: Smaller $\frac{\mathrm{d}S}{\mathrm{d}X}$ under MH
  - The markets alleviate the allocation conflict caused by MH.
  - In particular, the whole effect of MH is absorbed in the markets in this model, b/c PC binds under the linear contract.

Eqm riskless rate and MPRs: Moral hazard

$$
\begin{aligned}
r(t) &= r^s + \begin{pmatrix} \left(\gamma a \frac{e^{-\delta t}-e^{-\delta T}}{\delta}\right) \sum_{j=1}^{n} (\sigma_j^G)^2 \\ -\sum_{i=1}^{m} \left(1-(1+z_i^G)^{-\gamma}\right)\left(1-(1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}}\right)\lambda_i \end{pmatrix} \\
&= \delta + \gamma\mu^G + \left(\gamma a \frac{e^{-\delta t}-e^{-\delta T}}{\delta} - \frac{\gamma(\gamma+1)}{2}\right) \sum_{j=1}^{n} (\sigma_j^G)^2 \\
&\quad + \sum_{i=1}^{m} \left\{ \left(1-(1+z_i^G)^{-\gamma}\right)(1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}} - \gamma z_i^G \right\} \lambda_i, \\
\eta_j(t) &= \eta_j^s - a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\sigma_j^G \\
&= \left(\gamma - a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\right)\sigma_j^G \quad \text{for } j = 1, \cdots, n, \\
\xi_i(t) &= \xi_i^s + (1+z_i^G)^{-\gamma}\left(1-(1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}}\right) \\
&= 1-(1+z_i^G)^{-\gamma + a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}} \quad \text{for } i = 1, \cdots, m.
\end{aligned}
$$

Moral-hazard premium

[Moral-hazard premium]

$$
\begin{aligned}
r(t) - r^s \quad &= \quad \left(\gamma a \frac{e^{-\delta t} - e^{-\delta T}}{\delta}\right) \sum_{j=1}^{n} (\sigma_j^G)^2 \\
&\quad + \sum_{i=1}^{m} \left((1 + z_i^G)^{-\gamma} - 1\right) \left(1 - (1 + z_i^G)^{a \frac{e^{-\delta t} - e^{-\delta T}}{\delta}}\right) \lambda_i.
\end{aligned}
$$

[MH-driven distortions of the market prices of risk and jump risk]

$$
\begin{aligned}
\eta_j(t) - \eta_j^s \quad &= \quad -a \frac{e^{-\delta t} - e^{-\delta T}}{\delta} \sigma_j^G \leq 0 \\
&\quad \text{with equality if } t = T \quad \text{for } j = 1, \cdots, n, \\
\xi_i(t) - \xi_i^s \quad &= \quad (1 + z_i^G)^{-\gamma} \left(1 - (1 + z_i^G)^{a \frac{e^{-\delta t} - e^{-\delta T}}{\delta}}\right) \geq 0 \\
&\quad \text{with equality if } t = T \quad \text{for } i = 1, \cdots, m.
\end{aligned}
$$

Moral-hazard premium (cont.)

[Sketch of proof]:

$$\frac{\mathrm{d}\Lambda^*(t)}{\Lambda^*(t_-)} = \underbrace{-\delta\,\mathrm{d}t + \frac{\mathrm{d}\big(X(t)^{-\gamma}\big)}{X(t_-)^{-\gamma}}}_{\text{No moral-hazard case}} + \underbrace{\frac{\mathrm{d}Y(t)}{Y(t_-)}}_{\text{Distorted probability measure}} + \underbrace{\frac{\mathrm{d}Y(t)\,\mathrm{d}\big(X(t)^{-\gamma}\big)}{Y(t_-)X(t_-)^{-\gamma}}}_{\text{Quadratic covariation}}.$$

where

$$\frac{\mathrm{d}Y(t)}{Y(t_-)} = a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\sum_{j=1}^{n}\sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^{m}\left((1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}} - 1\right)\mathrm{d}M_i(t).$$

Recall $Y(t) = \mathbb{E}_t^{\mathbb{P}}\left[e^{a\int_0^T e^{-\delta u}\log X(u)\,\mathrm{d}u}\right]$: a time-$t$ twist of $\mathbb{Q}$ caused by X. □

Effect of non-stationarity: Perturbation via the Taylor series expansion

- Consider a modified contract form as follows: for $\hat{s} = s^*$ or s^{**},

$$S(t) = s^{\zeta}(t)X(t) \in \mathcal{S} \quad \text{with} \quad s^{\zeta}(t) := \hat{s}e^{\nu t + \varepsilon t^2}.$$

- We do not optimize s^{ζ} here, but impose the binding PC.

$$\begin{aligned}\nu &= -\frac{1}{\delta}\left(\frac{2 - e^{-\delta T}(2 + 2\delta T + \delta^2 T^2)}{1 - e^{-\delta T}(1 + \delta T)}\right)\varepsilon \\ &=: D(\delta, T)\varepsilon.\end{aligned}$$

- The effect of the perturbation:
  - The market prices of diffusive risk and jump risk are unchanged by the perturbation.
  - The drift term (i.e., the riskless rate) is influenced:

$$\frac{\mathrm{d}(1 - s^{\zeta}(t))^{-\gamma}}{(1 - s^{\zeta}(t))^{-\gamma}}\frac{1}{\mathrm{d}t} = -\gamma\frac{\mathrm{d}\log(1 - s^{\zeta}(t))}{\mathrm{d}t} = \frac{\gamma\hat{s}}{1 - \hat{s}}D(\delta, T)\varepsilon.$$

  - For large T, when $\varepsilon > 0$ (< 0), the effect is negative (positive).

Concluding remarks

- Solves explicitly for an equilibrium state prices (in particular, riskless rate/ market prices of risk and jump risk) under MH.
  - Prices any kinds of securities in the contingent-claims approach.
- Notably, shows that under the moral-hazard problem, a positive moral-hazard premium is stipulated on a riskless rate.
  - $\Rightarrow$ More serious risk-free rate puzzle under the MH problem.

Future work

- Limitations of this paper:
  - Power utility – time separable
    - SDU, Habit formation might resolve the risk-free rate puzzle (Nakamura et al. (2009)).
  - Representative model
    - Extension to a multi-firm model
    - Pricing profit loadings in insurance.
  - Stationary linear payment rule
    - W/O this, not easy to solve the problem even numerically, due to higher dimensional PDEs than three.
- Relaxing these assumptions will be our next work either analytically or numerically.
  - Our model could be a good benchmark for numerical approximations.

Go back to Contributions

Thank you! ☺

Appendices

Note on measure change

- A measure $\mathbb{Q}$ is absolutely continuous w.r.t. $\mathbb{P}$, written as $\mathbb{Q} \ll \mathbb{P}$, i.e., $\mathbb{P}(A) = 0$ implies $\mathbb{Q}(A) = 0$ for $A \in \mathcal{F}$.
- Define $Z(t) := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}(t)}$.
- By the Martingale Representation Theorem, $\exists$ $\mathbb{F}$-predicable processes θ_j and $\alpha_i \geq -1$ for all $i, j = 1$ such that

$$\mathrm{d}Z(t) = Z(t_-)\Big\{ \sum_{j=1}^{n} \theta_j(t)\,\mathrm{d}B_j(t) + \sum_{i=1}^{m} \alpha_i(t)\,\mathrm{d}M_i(t) \Big\}.$$

  - $\mathbb{Q}$-Brownian motion $\widetilde{B}_j(t) := B_j(t) - \int_0^t \theta_j(s)\,\mathrm{d}s$
  - $\mathbb{Q}$-(local) martingale $\widetilde{M}_i(t) := N_i(t) - \int_0^t \widetilde{\lambda}_i(s)\,\mathrm{d}s$ where $\widetilde{\lambda}_i(s) := \lambda_i\{\alpha_i(s) + 1\}$.

Go back to Moral hazard

Proof of Lemma

Taking exponential of $\mathbb{E}^{\mathbb{Q}}[\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u] - H(\mathbb{Q}\,||\,\mathbb{P})$,

$$
\begin{aligned}
& e^{\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u\right] - H(\mathbb{Q}\,||\,\mathbb{P})} = e^{\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u - \log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right]} \\
\leq \quad & \mathbb{E}^{\mathbb{Q}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u - \log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}}\right] \quad \text{(by Jensen's inequality)} \\
= \quad & \mathbb{E}^{\mathbb{Q}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u} \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\right] = \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u} \mathbf{1}_{\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} > 0\}}\right] \\
\leq \quad & \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}\right]
\end{aligned}
$$

with equality if and only if $\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u - \log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ is a constant. Therefore,

$$
\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \frac{e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}}{\mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}\right]}.
$$

Thus $\mathbb{Q}^*$ is obtained. □

Go back to Firm's optimization

Value function in market equilibrium $J^*(t, X(t), Y(t))$

- $J^*(t, X(t), Y(t)) := J(t, X(t), Y(t), 0)$.
- Since $W^*(t) = 0 \, \forall \, t$ a.s.,

$$
\begin{aligned}
0 &= K(1-s^*)^{1-\gamma} y \frac{x^{1-\gamma}}{1-\gamma} + J_t^* + \mu^G x J_x^* \\
&+ \frac{1}{2} \sum_{j=1}^{n} \left\{ (\sigma_j^G)^2 x^2 J_{xx}^* + (a \frac{e^{-\delta t} - e^{-\delta T}}{\delta} \sigma_j^G)^2 y^2 J_{yy}^* \right\} \\
&+ \sum_{j=1}^{n} a \frac{e^{-\delta t} - e^{-\delta T}}{\delta} (\sigma_j^G)^2 x y J_{xy}^* \\
&+ \sum_{i=1}^{m} \lambda_i \left\{ J^*(t, (1+z_i^G)x, (1+z_i^G)^{a \frac{e^{-\delta t} - e^{-\delta T}}{\delta}} y) - J^*(t, x, y) \right\}
\end{aligned}
$$

Value function (cont.)

Try $J^*(t,x,y) = py\frac{x^{1-\gamma}}{1-\gamma}$ with $p = p(t)$ with $p(T) = 0$:

$$p' + pL(t) = -K(1-s^*)^{1-\gamma}; \quad p(T) = 0.$$

where

$$\begin{aligned} L(t) \quad &:= \quad (1-\gamma)\mu^G - (1-\gamma)\sigma^G(\sigma^G)^\top \left(\frac{\gamma}{2} - a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\right) \\ &\quad + \sum_{i=1}^{m} \lambda_i \left\{(1+z_i^G)^{(1-\gamma)+a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}} - 1\right\}. \end{aligned}$$

Value function (cont.)

From $p(T) = 0$,

$$p(t) = \frac{K(1-s^*)^{1-\gamma}}{L(t)} \left(\frac{L(t)\exp\left\{-\int L(t)\,\mathrm{d}t\right\}}{L(T)\exp\left\{-\left(\int L(t)\,\mathrm{d}t\right)_{t=T}\right\}} - 1 \right)$$

where

$$\begin{aligned} L(T) &= (1-\gamma)\mu^G - \frac{\gamma(1-\gamma)}{2}\sigma^G(\sigma^G)^\top + \sum_{i=1}^m \lambda_i \left\{(1+z_i^G)^{1-\gamma} - 1\right\}, \\ \left(\int L(t)\,\mathrm{d}t\right)_{t=T} &= \left((1-\gamma)\mu^G - (1-\gamma)\sigma^G(\sigma^G)^\top \left(\frac{\gamma}{2} + \frac{a}{\delta}e^{-\delta T}\right) - \sum_{i=1}^m \lambda_i\right) T \\ &\quad -(1-\gamma)\sigma^G(\sigma^G)^\top \frac{a}{\delta^2}e^{-\delta T} - \frac{1}{(2-\gamma)ae^{-\delta T}} \sum_{i=1}^m \lambda_i (1+z_i^G)^{2-\gamma}. \end{aligned}$$

Also,

$$\mu^R + z^R\lambda = -\sigma^R(\sigma^G)^\top \left(-\gamma + a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\right).$$

Go back to Optimality and equilibrium

Derivation of Equilibrium state price: (1) State price

- Look at ex-post optimal allocation, taking S^* as given.
- Define a consumption/wealth process as

$$\phi(t) := \begin{cases} C(t) & \text{for } 0 \le t < T \\ W(T) & \text{for } t = T \end{cases}$$

- Define $\Phi(S^*)$ as the subset of $\mathcal{A}$ consisting of (ϕ, β) given S^*.
- Define $\Pi \in \mathcal{H}_+$ for some market prices of (diffusive) risk and jump risk η, ξ as:

$$\mathrm{d}\Pi(t) = \Pi(t_-)\Big(- r(t)\,\mathrm{d}t - \eta(t)^\top \mathrm{d}B(t) - \xi(t)^\top \mathrm{d}M(t)\Big), \quad \Pi(0) = 1.$$

Definition

Π is said to be a state price at $(\phi, \beta) \in \Phi(S^*)$ if, for $(\phi, \beta) \in \Phi(S^*)$ and for any $h \in \mathcal{H}$ such that $(\phi + h, \beta') \in \Phi(S^*)$, $(\Pi|h) \le 0$.

(1) State price (cont.)

Assumption

1. $\mu^R = \sigma^R \eta + z^R I^\xi \lambda$.
2. *For* $(\phi, \beta) \in \Phi(S^*)$, $\mathbb{E}^{\mathbb{P}}[\sup_t \Pi(t) W(t)] < \infty$.

Lemma

Π *is a state price at* $(\phi, \beta) \in \Phi(S^*)$.

[Sketch of proof]: By Lebesgue's dominated convergence theorem,

$$\begin{aligned} w_0 &= \Pi(0)W(0) \\ &= \mathbb{E}^{\mathbb{P}}\left[\int_0^T \Pi(u)(C(u) + S^*(u) - X(u))\,\mathrm{d}u + \Pi(T)W(T)\right] = (\Pi|\phi + S^* - X). \end{aligned}$$

On the other hand, for any $h \in \mathcal{H}$ such that $\phi + h \in \Phi(S^*)$, by Fatou's lemma,

$$\Pi(0)W(0) + (\Pi|X - S^*) \geq (\Pi|\phi + h).$$

Therefore, $(\Pi|h) \leq 0$. $\square$

(2) Utility maximization

- Redefintion: (1) Under $S = S^*$ AND (2) When PC binds.
  - $\widehat{\mathbb{Q}}^*$: the optimal probability measure
  - $\{\widehat{U}_2(t)\}_{0 \le t \le T}$: the investor's utility process
- By the Martingale Representation Theorem,

$$\begin{aligned} \mathrm{d}\widehat{U}_2(t) &= -\widehat{F}(t, \phi(t), \widehat{U}_2(t); \widehat{\mathbb{Q}}^*)\,\mathrm{d}t + \Sigma(t)\,\mathrm{d}B(t) + \Gamma(t)\,\mathrm{d}M(t), \\ \widehat{U}_2(T) &= \widehat{F}(T, \phi(T), \widehat{U}_2(T);\, \widehat{\mathbb{Q}}^*) \end{aligned}$$

where

$$\widehat{F}(t, \phi(t), \widehat{U}_2(t); \widehat{\mathbb{Q}}^*) := \begin{cases} e^{-\rho} e^{\frac{1-e^{-\delta T}}{\delta} a \log s^*} Y(t) \frac{C(t)^{1-\gamma}}{1-\gamma} - \delta \widehat{U}_2(t) & \text{for } 0 \le t < T, \\ e^{-\rho} e^{\frac{1-e^{-\delta T}}{\delta} a \log s^*} Y(T) W(T) & \text{for } t = T. \end{cases}$$

(2) Utility maximization (cont.)

Define $\mathcal{E}(t) := e^{-\delta t}$ and

$$\Lambda(t) := \mathcal{E}(t)\widehat{F}_\phi(t) := \begin{cases} \mathcal{E}(t)\widehat{F}_c(t) & \text{for } t \in [0, T), \\ \mathcal{E}(T)e^{-\rho} e^{\frac{1-e^{-\delta T}}{\delta} a \log s^*} Y(T) & \text{for } t = T. \end{cases}$$

where $\widehat{F}_c(t) = e^{-\rho} e^{\frac{1-e^{-\delta T}}{\delta} a \log s^*} Y(t) C(t)^{-\gamma}$.

Lemma

For $(\phi, \beta) \in \Phi(S^*)$ and for any $h \in \mathcal{H}$ such that $(\phi + h, \beta') \in \Phi(S^*)$,

$$\widehat{U}_2(\phi + h, S^*) \leq \widehat{U}_2(\phi, S^*) + (\Lambda | h).$$

(2) Utility maximization (cont.)

[Sketch of proof]: Define

$$\begin{aligned}
\Delta_U &:= \widehat{U}_2(\phi + h, S^*) - \widehat{U}_2(\phi, S^*), \\
\Delta_\Sigma &:= \Sigma(\phi + h, S^*) - \Sigma(\phi, S^*), \\
\Delta_\Gamma &:= \Gamma(\phi + h, S^*) - \Gamma(\phi, S^*), \\
\Delta_F(t) &:= \begin{cases} \widehat{F}(\phi, S^*, \widehat{U}_2) + \widehat{F}_c h - \delta\Delta_U - \widehat{F}(\phi + h, S^*, \widehat{U}_2 + \Delta_U) & \text{for } 0 \le t < T, \\ 0 & \text{for } t = T. \end{cases}
\end{aligned}$$

Then,

$$\begin{aligned}
\mathrm{d}\Delta_U &= \mathrm{d}\widehat{U}_2(\phi + h, S^*) - \mathrm{d}\widehat{U}_2(\phi, S^*) \\
&= -(\widehat{F}_c h - \delta\Delta_U - \Delta_F)\,\mathrm{d}t + \Delta_\Sigma\,\mathrm{d}B + \Delta_\Gamma\,\mathrm{d}M; \\
\Delta_U(T) &= \widehat{F}_\phi(T)h(T) - \Delta_F(T) = \widehat{F}_\phi(T)h(T).
\end{aligned}$$

By Lebesgue's dominated convergence theorem,

$$\begin{aligned}
\mathcal{E}(0)\Delta_U(0) &= \mathbb{E}^{\mathbb{P}}\left[\int_0^T (\mathcal{E}\widehat{F}_c h - \mathcal{E}\Delta_F)\,\mathrm{d}t + \Big(\mathcal{E}(T)\widehat{F}_\phi(T)h(T) - \mathcal{E}(T)\Delta_F(T)\Big)\right] \\
&= (\mathcal{E}\widehat{F}_\phi | h) - (\mathcal{E}|\Delta_F).
\end{aligned}$$

Since $\Delta_F(t) \ge 0 \,\forall\, t$, $\widehat{U}_2(\phi + h, S^*) - \widehat{U}_2(\phi, S^*) \le (\Lambda | h)$. $\square$

Go back to Equilibrium state price