

Unit 3. Elasticity

Learning objectives

To comprehend and apply the concepts of elasticity, including calculating:

- price elasticity of demand;
- cross-price elasticity of demand;
- income elasticity of demand;
- price elasticity of supply.

Questions for revision:

- ✓ Demand schedule and the law of demand;
- ✓ Factors of demand, complements and substitutes;
- ✓ Supply schedule and the law of supply;
- ✓ Market equilibrium: welfare aspects of government controls.

3.1. Price elasticity of demand

Price elasticity of demand is the measure of intensity of reaction of quantity demanded in response to a change in price. It shows a percentage change in quantity demanded that results from one percent change in price:

$$E_p^d = \frac{(Q_2 - Q_1) / (P_2 - P_1)}{Q_1 / P_1} = \frac{\Delta Q / \Delta P}{Q / P} = \frac{\Delta Q}{\Delta P} \cdot \frac{P}{Q}$$

This is the point price elasticity of demand in the form of discrete changes in price and quantity demanded. If the changes in price and quantity demanded are infinitesimal, one can calculate price elasticity of demand in differential form as the limiting case of the discrete ratio:

$$E_p^d = \lim_{\Delta P \rightarrow 0} \left(\frac{\Delta Q}{Q} / \frac{\Delta P}{P} \right) = \lim_{\Delta P \rightarrow 0} \left(\frac{\Delta Q}{\Delta P} \cdot \frac{P}{Q} \right) = \frac{dQ}{dP} \cdot \frac{P}{Q}$$

To eliminate the indeterminance of the starting point of change that gives the reference price and quantity for the corresponding changes in the variables one can calculate arc price elasticity of demand where average values of the variables serve as the base for relative changes:

$$E_p^d = \frac{(Q_2 - Q_1) / (P_2 - P_1)}{(Q_2 + Q_1) / 2 / (P_2 + P_1) / 2} = \frac{(Q_2 - Q_1) / (P_2 - P_1)}{(Q_2 + Q_1) / (P_2 + P_1)}$$

Mind the difference between elasticity and the slope of demand curve. For instance, despite the slope of the demand curve D_a in the figure 3.1 is twice as much as the slope of the curve D_b , taking into consideration

that $\frac{dQ}{dP} = tg\alpha$ for the curve D_a and $\frac{dQ}{dP} = tg\beta$ for the curve D_b at the price P^* elasticity of demand is one and the same for both curves:

$$E_p^{D_b} = tg\beta \cdot \frac{P^*}{Q_b} = \frac{2k}{n} \cdot \frac{P^*}{Q_b} = 2tg\alpha \cdot \frac{P^*}{2Q_a} = tg\alpha \cdot \frac{P^*}{Q_a} = E_p^{D_a}.$$

And vice versa, price elasticity may be different despite equal slopes of linear demand curves. For instance, in the figure 3.2 at the given price P^* quantity of demand for D_b is twice as much as for D_a : $Q_b = 2Q_a$. The lines D_a and D_b are parallel. It implies that at the given points the curve D_a is twice as much elastic as D_b :

$$E_p^{D_b} = tg\beta \cdot \frac{P^*}{Q_b} = tg\alpha \cdot \frac{P^*}{2Q_a} = \frac{E_p^{D_a}}{2}.$$

Figure 3.1.
Equal elasticities and
different slopes of demand
curves

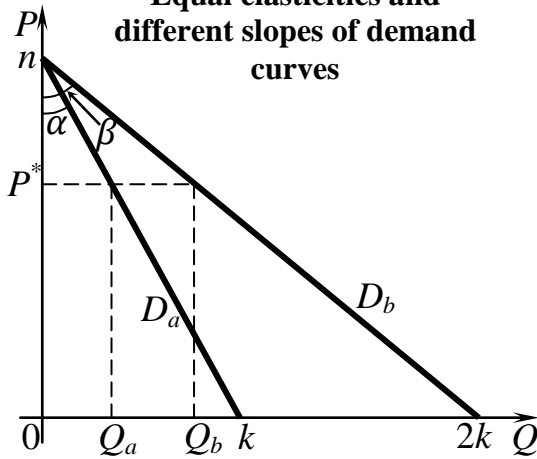
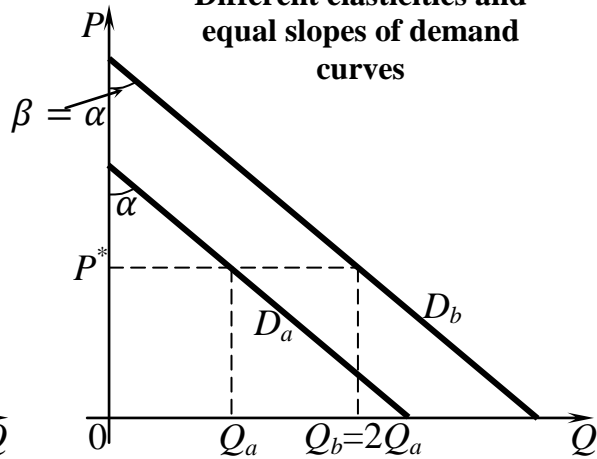


Figure 3.2.
Different elasticities and
equal slopes of demand
curves



The reason to use elasticity instead of derivative of a function

$y = f(x)$ is that elasticity $E_x^y = \frac{dy}{dx} \cdot \frac{x}{y}$ is a relative ratio that is independent of numeraires of the variables – y and x :

$$E_{ax}^{by} = \frac{d(by)}{d(ax)} \cdot \frac{ax}{by} = \frac{b dy}{a dx} \cdot \frac{ax}{by} = \frac{dy}{dx} \cdot \frac{x}{y} = E_x^y.$$

Using the derivative of logarithm of quantity demanded $\frac{d \ln Q}{dQ} = \frac{1}{Q}$,
i.e. $\frac{dQ}{Q} = d \ln Q$, and the derivative of logarithm of market price $\frac{d \ln P}{dP} = \frac{1}{P}$,
i.e. $\frac{dP}{P} = d \ln P$, one can calculate price elasticity of demand as the ratio of
differentials of the corresponding logarithms: $E_p^d = \frac{dQ}{dP} \cdot \frac{P}{Q} = \frac{dQ}{Q} / \frac{dP}{P} = \frac{d \ln Q}{d \ln P}$.

Demand is said to be elastic if $E_p^d < -1$. Demand is said to be inelastic if $-1 < E_p^d < 0$. Demand is said to be unit elastic if $E_p^d = -1$. We suppose here that the law of demand holds.

3.2. Elasticity of demand and total revenue of producers

Total revenue of producers as a function of output can be expressed using price elasticity of demand:

$$\begin{aligned} MR &= \frac{dTR}{dQ} = \frac{d}{dQ} (p(Q)Q) = p(Q) + \frac{dp(Q)}{dQ} Q \\ &= p(Q) \left(1 + \frac{dp(Q)}{dQ} \cdot \frac{Q}{p(Q)} \right) = p(Q)(1 + E_q^d) \\ &= p \left(1 + \frac{1}{\frac{dQ(p)}{dp} \cdot \frac{p}{Q(p)}} \right) = p \left(1 + \frac{1}{E_p^d} \right), \end{aligned}$$

where $MR = \frac{dTR}{dQ}$ is marginal revenue of producer(s), $E_p^d = \frac{dQ(p)}{dp} \cdot \frac{p}{Q(p)}$ – price elasticity of demand, $E_q^d = \frac{dp(Q)}{dQ} \cdot \frac{Q}{p(Q)}$ – elasticity of quantity demanded.

Note that derivatives of inverse functions $y = f(x)$ and $x = f^{-1}(y)$ are inverse values: $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$. The same applies to elasticities of

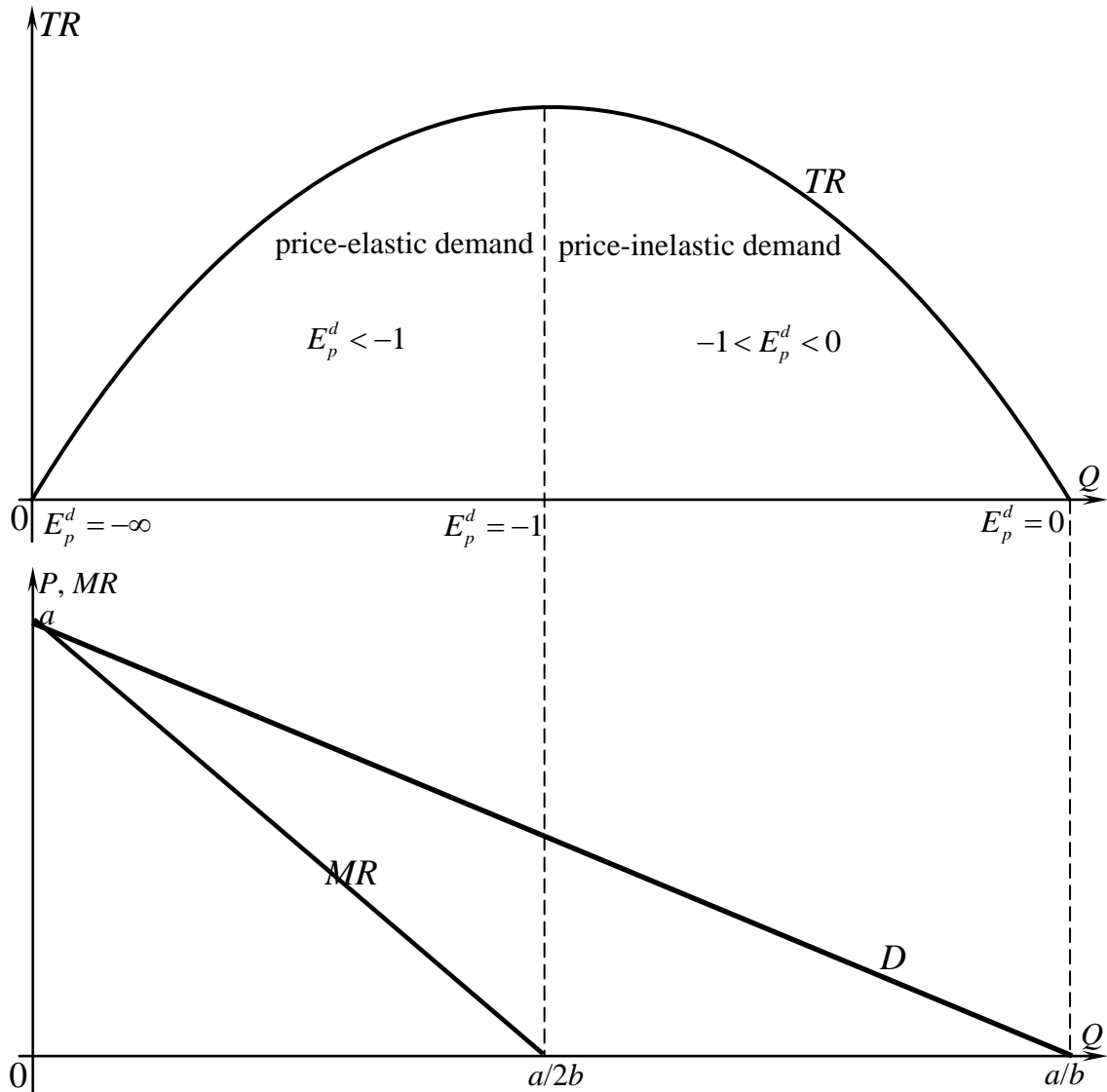
inverse functions – they are inverse values: $E_x^y = \frac{dy}{dx} \cdot \frac{x}{y} = \frac{1}{\frac{dx}{dy} \cdot \frac{y}{x}} = \frac{1}{E_y^x}$.

In case of elastic demand ($E_p^d < -1$) $\frac{dTR}{dQ} > 0$. It implies that total revenue is an increasing function of the quantity of the good produced: when Q goes up, TR grows as well; and a reduction in Q yields a decline in TR . If demand is inelastic, i.e. $-1 < E_p^d < 0$, $\frac{dTR}{dQ} < 0$, and total revenue is a decreasing function of output: when Q goes up, TR declines; when Q goes down, TR grows. In case of unit elasticity of demand $\frac{dTR}{dQ} = 0$, and total revenue is at the maximum.

In the figure below demand is assumed to be linear: $P = a - bQ$, where a and b are constant numbers. It implies that total revenue is parabola $TR = P \cdot Q = aQ - bQ^2$, and marginal revenue is linear and twice as steep as the demand curve with the same intercept a at the vertical axes: $MR = TR' = a - 2bQ$.

A typical relationship between total revenue of producers and price elasticity of linear demand $P = a - bQ$, or $Q = \frac{a}{b} - \frac{P}{b}$, is depicted below.

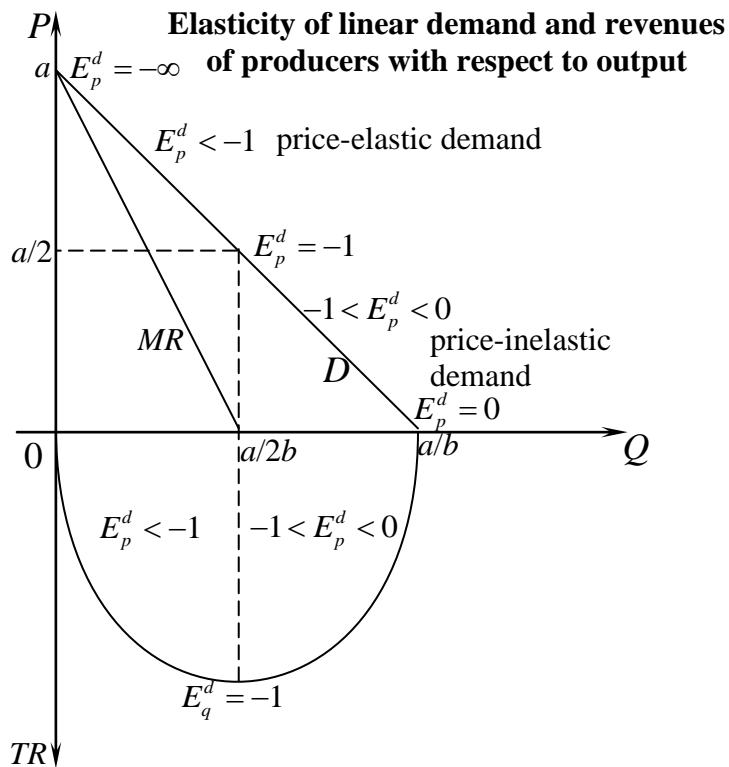
Elasticity of linear demand and revenues of producers as related to output



Use the definition of elasticity of quantity demanded for linear demand curve $P = a - bQ$ to get: $E_q^d = \frac{dP}{dQ} \cdot \frac{Q}{P} = -\frac{bQ}{a-bQ}$. Elasticities of inverse functions are. Price elasticity of inverse linear demand curve $Q = \frac{a}{b} - \frac{P}{b}$ is an inverse value of elasticity of quantity demanded E_q^d :

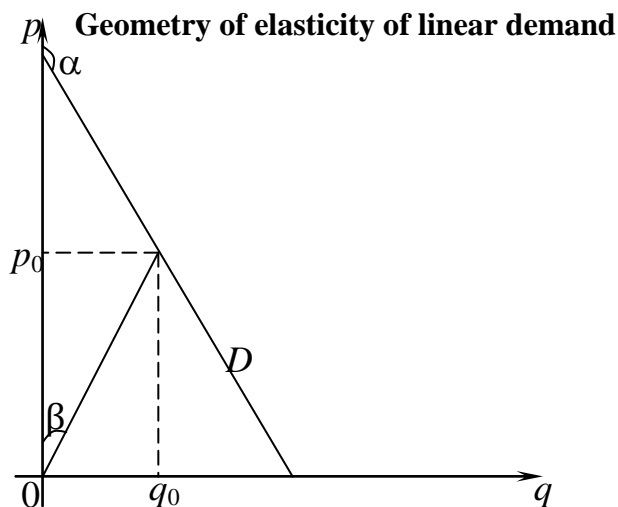
$$E_p^d = -\frac{a - bQ}{bQ} = -\frac{P}{b\left(\frac{a}{b} - \frac{P}{b}\right)}$$

For the reasons that will be clarified later let's rotate the total revenue curve and plot it in the direction to the bottom of the graph.



Elasticity of linear demand is equal to the ratio of its slope (with respect to P axis) and the slope of the ray coming from the origin and going through the given point (p_0, q_0) on the demand curve: $E_p^d = \frac{tg\alpha}{tg\beta}$. The slope of linear demand curve ($tg\alpha$) is constant, and ratio $\frac{q}{p} = tg\beta$ goes up with Q (see the figure below). So, elasticity of linear demand curve is not constant, its absolute value declines with Q :

$$E_p^d = \frac{dQ}{dP} \cdot \frac{P}{Q} = -\frac{P}{b\left(\frac{a}{b} - \frac{P}{b}\right)} = -\frac{P}{a - P} = -\frac{a - bQ}{bQ} = 1 - \frac{a}{bQ}$$

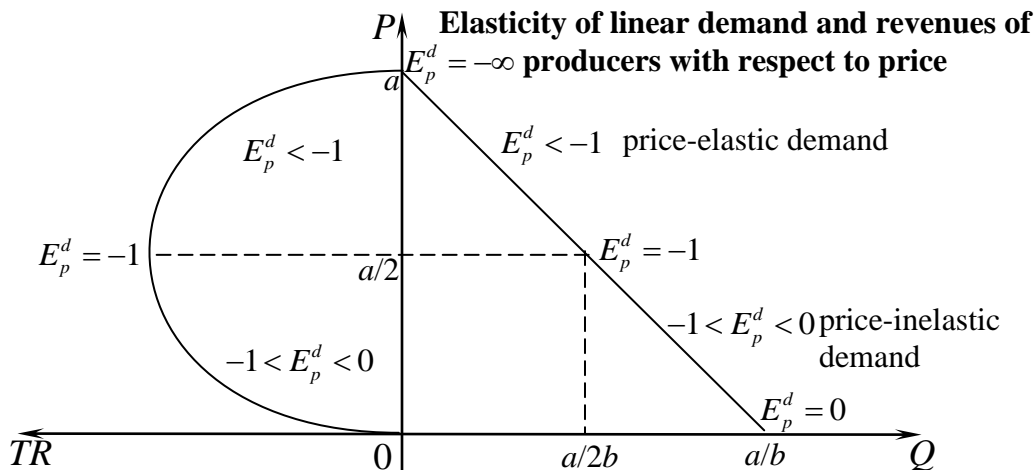


Quantity of the good $0 < Q < a/2b$ corresponds to price elastic demand, and the range $a/2b < Q < a/b$ – to price inelastic demand. In the midpoint ($Q = a/2b$) linear demand is unit-elastic.

There is the following relationship between total revenue of producers which is treated as a function of market price and price elasticity of demand:

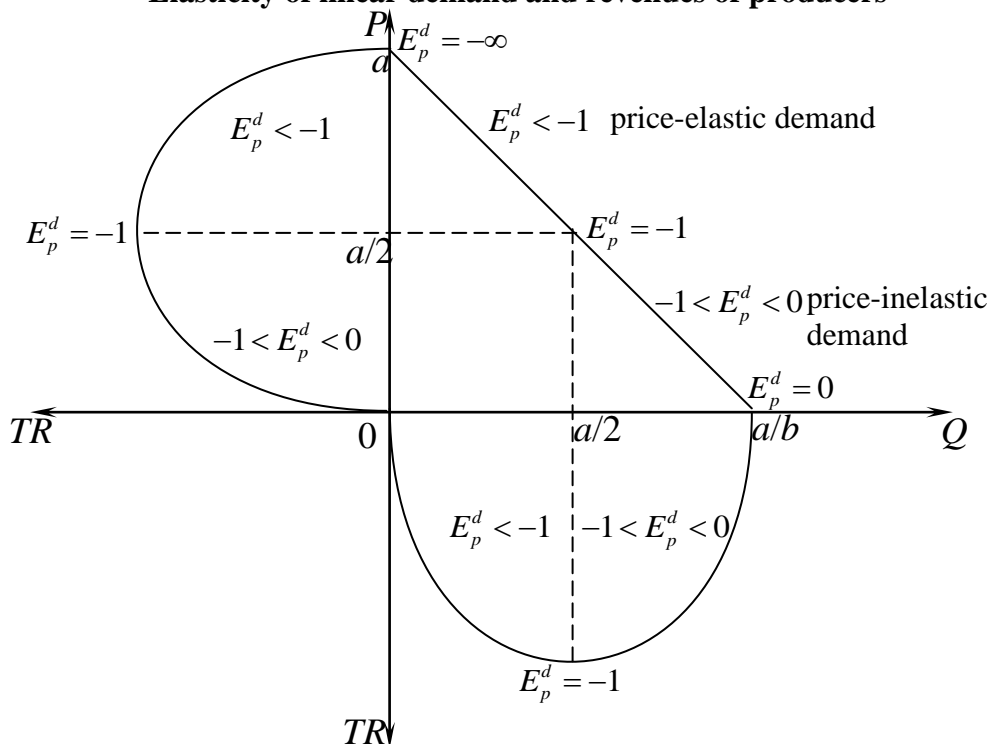
$$\begin{aligned} \frac{dTR}{dp} &= \frac{d}{dp}(Q(p)p) = Q(p) + \frac{dQ(p)}{dp}p = Q(p) \left(1 + \frac{dQ(p)}{dp} \cdot \frac{p}{Q(p)} \right) \\ &= Q(p)(1 + E_p^d) = Q(p) \left(1 + \frac{1}{\frac{dp(Q)}{dQ} \cdot \frac{Q}{p(Q)}} \right) \\ &= p \left(1 + \frac{1}{E_q^d} \right). \end{aligned}$$

If demand is price elastic, $\frac{dTR}{dp} < 0$. A price increase will reduce total expenditure, and a price reduction will increase total expenditure. If demand is price inelastic, $\frac{dTR}{dp} > 0$. In this case a price increase will push total expenditure up, and a price reduction will decrease total expenditure. Total expenditure is highest when demand is unit elastic (see the figure below).



Combine the graph with the similar figure above to get a joint illustration of dependence between elasticity of linear demand and total revenue of producers (see the figure below).

Elasticity of linear demand and revenues of producers



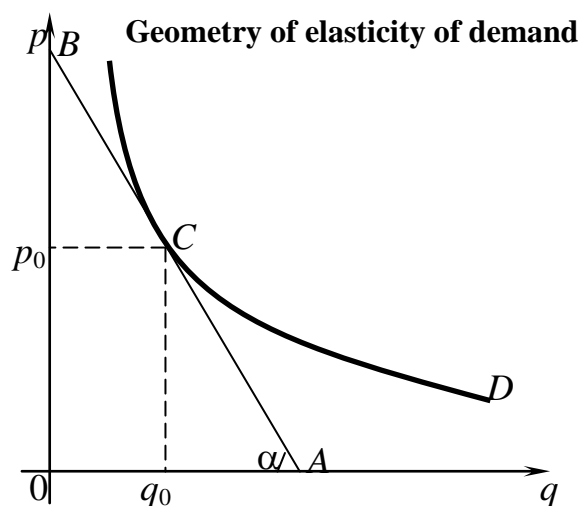
Note that price elasticity of power demand curve $Q = P^\alpha$ is constant and is equal to the power α :

$$E_p^d = \frac{dQ}{dP} \cdot \frac{P}{Q} = \frac{dP^\alpha}{dP} \frac{P}{P^\alpha} = \alpha P^{\alpha-1} \frac{P}{P^\alpha} = \alpha.$$

For instance, in the constant revenue case ($TR=PQ=const$), when demand curve is hyperbola: $Q = \frac{1}{P}$, it is unit-elastic regardless of quantity demanded (or price).

Price elasticity of demand can be calculated as the negative of the ratio of segments of a line tangent to the demand curve from the point of tangency to the horizontal and vertical axis correspondingly: $E_p^d = -\frac{CA}{CB}$ (see the figure below).

Proof: $Aq_0 = \frac{p_0}{tg\alpha}$, where $tg\alpha = -\frac{dp}{dq}$. It implies: $Aq_0 = -\frac{p_0}{\frac{dp}{dq}} = -\frac{dq}{dp}p_0$. Taking into consideration similarity of triangles: $\Delta CBp_0 \sim \Delta ACq_0$, we get: $\frac{AC}{CB} = \frac{Aq_0}{q_0} = -\frac{dq}{dp} \frac{p_0}{q_0} = -E_p^d$, q.e.d.



In a linear case demand schedule will coincide with its tangent line. Thus, moving along the demand curve we observe a decline of absolute value of elasticity from ∞ to 0.

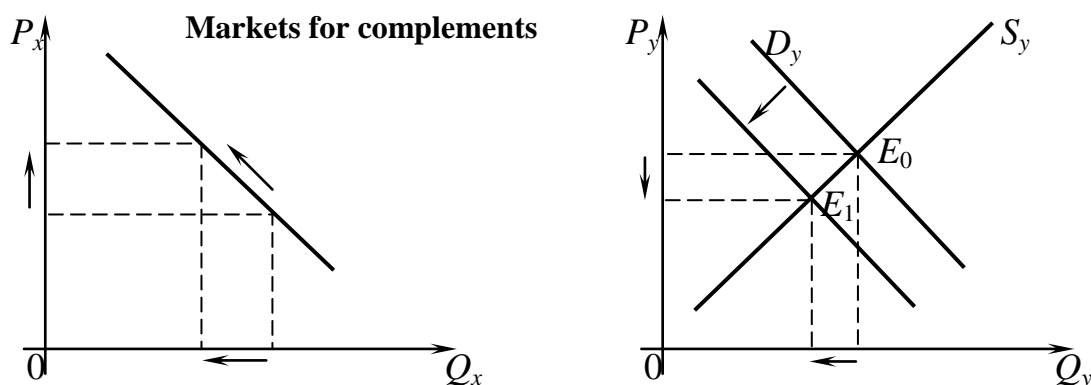
3.3. Cross-price elasticity of demand. Income elasticity of demand

Cross-price elasticity of demand is the ratio that shows a percentage change of quantity demanded of a good in response to one percent change in the price of the other:

$$E_{p_y}^{d_x} = \frac{(Q_2^x - Q_1^x) / Q_1^x}{(P_2^x - P_1^x) / P_1^x} = \frac{\Delta Q_x / Q_x}{\Delta P_y / P_y} = \frac{\Delta Q_x}{\Delta P_y} \cdot \frac{P_y}{Q_x}$$

By definition the goods are complements if their quantities change in one and the same direction: when quantity of a good goes up consumers increase the demand for the other one. And vice versa, a decrease in demand for a good pushes down consumption of the other one. Under the law of demand an upward movement of the price of a good x tends to decrease its consumption. As the good y is complementary for x , it shifts demand curve for y inwards (towards the origin), and both its quantity and price go down (see the figure below). It follows that an increase in the price of a good pushes down consumption of its complement, so cross-price elasticity of demand for complementary goods is negative, as a rule¹.

¹ The exception is the case of a Giffen good. It will be considered later, in unit 4 “Consumer choice”.



If the goods are substitutes, an increase in price of a good results in an increase in price of the other one: $P_x \uparrow \rightarrow Q_x \downarrow \rightarrow Q_y \uparrow$ (see the corresponding figure in unit 2). Consequently, cross-price elasticity of substitutes is positive².

Income elasticity of demand is the ratio that shows a percentage change of quantity demanded of a good in response to one percent change in income of consumers:

$$E_I^d = \frac{\Delta Q}{Q} / \frac{\Delta I}{I} = \frac{\Delta Q}{\Delta I} \cdot \frac{I}{Q}$$

If the good is normal, an increase in consumers' income ($\frac{\Delta I}{I} > 0$) yields an increase in the quantity demanded ($\frac{\Delta Q}{Q} > 0$). Consequently, income elasticity of demand for normal goods is positive: $E_I^d = \frac{\Delta Q}{Q} / \frac{\Delta I}{I} > 0$. If the good is inferior, an increase in consumers' incomes ($\frac{\Delta I}{I} > 0$) yields a fall in quantity demanded ($\frac{\Delta Q}{Q} < 0$). It follows that income elasticity of demand for inferior goods is negative: $E_I^d = \frac{\Delta Q}{Q} / \frac{\Delta I}{I} < 0$.

3.4. Price elasticity of supply

Price elasticity of supply is the percentage change in quantity supplied that occurs in response to a 1 percent change in price:

$$E_p^s = \frac{(Q_2 - Q_1)}{Q_1} / \frac{(P_2 - P_1)}{P_1} = \frac{\Delta Q}{Q} / \frac{\Delta P}{P} = \frac{\Delta Q}{\Delta P} \cdot \frac{P}{Q}$$

This is the expression of price elasticity of supply in case of discrete, finite changes in price and quantity of supply. If the changes in price and quantity

² The more precise treatment is postponed until unit 4 "Consumer choice".

of supply are infinitesimal one can use marginal analysis to calculate price elasticity of supply:

$$E_p^S = \lim_{\Delta P \rightarrow 0} \left(\frac{\Delta Q}{Q} / \frac{\Delta P}{P} \right) = \lim_{\Delta P \rightarrow 0} \left(\frac{\Delta Q}{\Delta P} \cdot \frac{P}{Q} \right) = \frac{dQ}{dP} \cdot \frac{P}{Q}$$

3.5. Application of elasticity theory: tax burden of producers and consumers

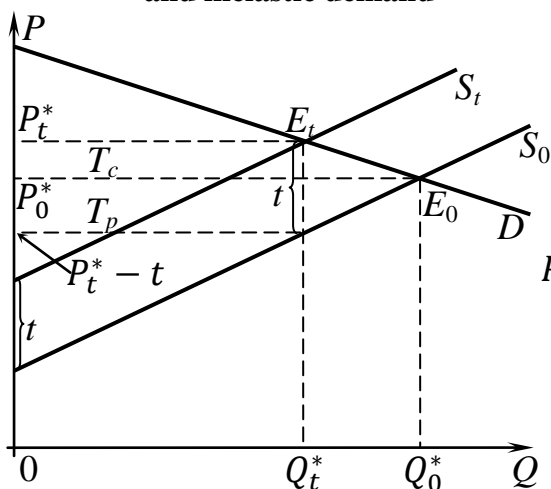
Incidence of a tax describes who eventually bears the burden of it. Let's calculate tax burden of consumers and producers to prove that it depends on the relative elasticities of demand and supply: $T = tQ_t^* = T_c + T_p$, $T_c = Q_t^*(P_t^* - P_0^*)$, $T_p = Q_t^*(P_0^* - P_t^* + t)$, where T_c is the tax burden of consumers, T_p – tax burden of producers, $T = T_c + T_p$ is the total tax revenue of the government (see the two figures below).

Use the expressions of elasticities of demand and supply to get:

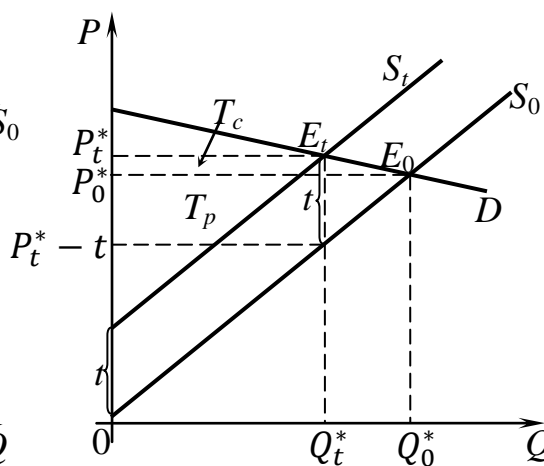
$$\frac{T_c}{T_p} = \frac{Q_t^*(P_t^* - P_0^*)}{Q_t^*(P_0^* - P_t^* + t)} = - \frac{\frac{(Q_t^* - Q_0^*)/Q_0^*}{(P_t^* - t - P_0^*)/P_0^*}}{\frac{(Q_t^* - Q_0^*)/Q_0^*}{(P_t^* - P_0^*)/P_0^*}} = - \frac{E_p^S}{E_p^D}$$

The relative tax burden of consumers and producers is the inverse ratio of absolute values of corresponding elasticities, i.e. to the negative of the ratio of elasticities of supply and demand: $\frac{T_c}{T_p} = - \frac{E_p^S}{E_p^D}$. The more elastic demand and the less elastic supply curves are the greater is the share of the tax levied on producers as compared to that of consumers (see the two figures below).

Elasticity of supply and demand and tax burden of consumers and producers: relatively elastic supply and inelastic demand



Elasticity of supply and demand and tax burden of consumers and producers: relatively inelastic supply and elastic demand



Let's go on with the examples of government fiscal policies in the unit 2 "Supply and demand". Suppose at first that market demand and supply curves are given by the following expressions: $Q_d = 100 - P$; $Q_s = 3P - 20$. If the government imposes a tax equal to 8 dollars per unit sold by producers total tax revenues of the government will be $T=8 \cdot 64=512$; and the shares of tax paid by consumers and producers are: $T_c=(36-30)64=384$ (75%), $T_p=(30-28)64=128$ (25%) (see the left hand side of the last but one figure in the unit 2 "Supply and demand").

If the government imposes ad valorem (sales) tax equal to 15% of producers' revenues total tax revenues of the government will be: $T=0,15 \cdot 66,2 \cdot 33,8=335,634$; and the shares of tax paid by consumers and producers are: $T_c=(33,8-30)66,2=251,56$ (75%); $T_p=(30-28,7)66,2=86,06$ (25%) (see the right hand side of the last but one figure in the unit 2 "Supply and demand").